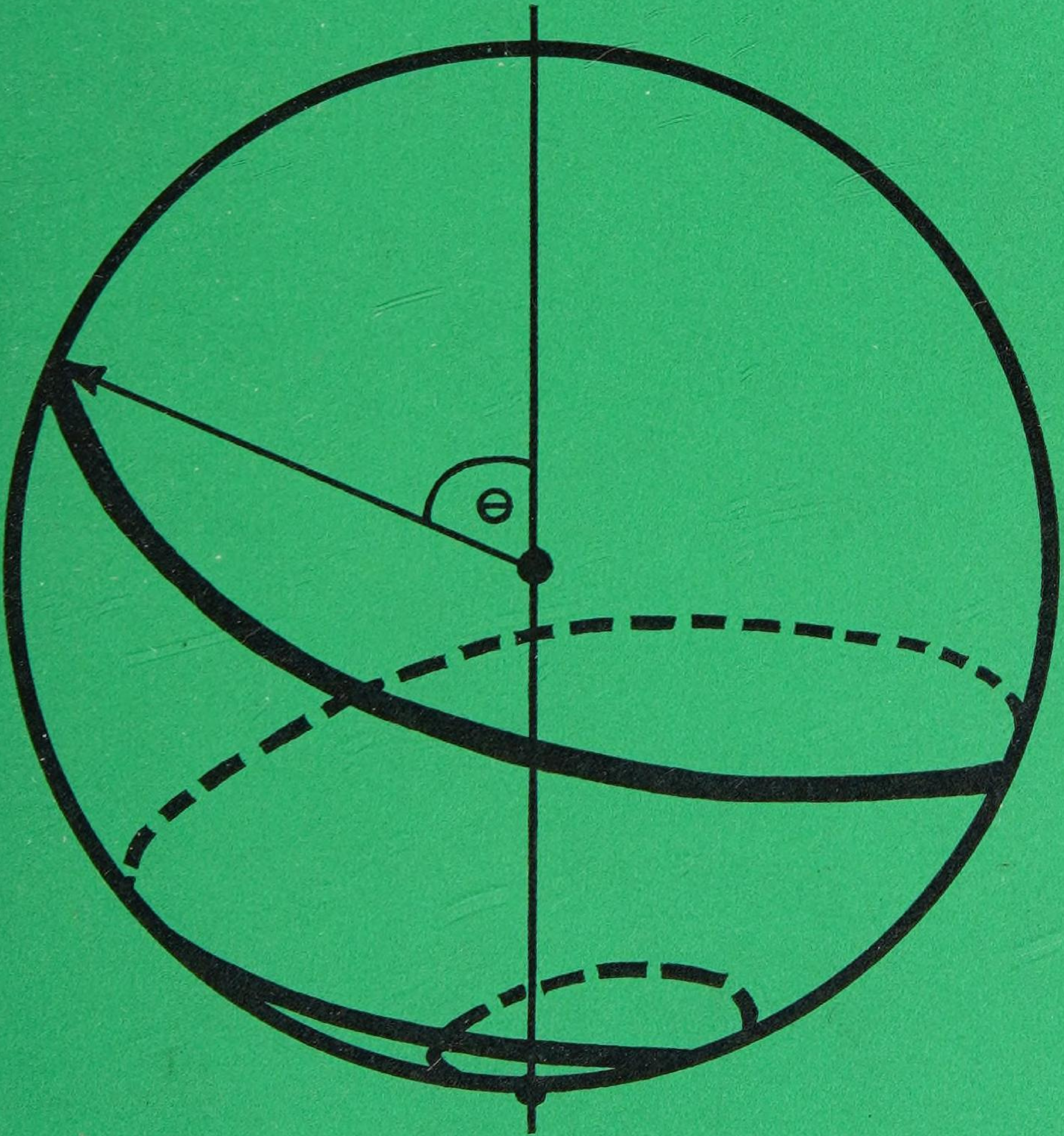
Magnetism and Magnetic Materials

J. C. ANDERSON



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Magnetism is one of the subjects in which considerable advances have been made, in both theoretical understanding and experimental work, over the last decade, and the tremendous strides in magnetic materials may be expected to continue, with a large number of new and sophisticated applications in computers and the aero-space industry. A result of this rapid advancement has been the appearance of textbooks of ever-increasing size and specialization. This compact book provides a fundamental treatment of the whole subject, which will equip the reader to understand the new developments. It is intended principally for electrical engineers and physicists; however, it is hoped that it will prove useful to anyone trained in a scientific discipline that includes a reasonable grasp of the necessary mathematics. An elementary knowledge of vector algebra and of differential and integral calculus is assumed.

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Magnetism and Magnetic Materials

MODERN ELECTRICAL STUDIES

A Series Edited by
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Head of Department of Electronics
University of Southampton

Magnetism and Magnetic Materials

J.C. ANDERSON

D.Sc.(Eng.), Ph.D., D.I.C., M.I.E.E., M.I.E.R.E.
C.Eng., F.Inst.P.

Professor of Electrical Materials, Imperial College, London

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Preface

Magnetism is one of the subjects in which considerable advances have been made, both in theoretical understanding and in experimental work, over the last decade. A result of this has been the appearance of textbooks of ever-increasing size and specialization. It is the intention of this book to provide a fundamental treatment of the subject in a form, and with a coverage, suitable for undergraduates in their penultimate and final years and for postgraduates embarking on study and research in some branch of magnetism. The book employs Standard International (m.k.s.) units throughout and is intended principally for electrical engineers and physicists. It is hoped, however, that it will prove useful to anyone trained in a scientific discipline that includes a reasonable grasp of the necessary mathematics. An elementary knowledge of vector algebra and of differential and integral calculus is assumed.

Chapter 1 is devoted to the basic concepts and laws of magnetism giving derivations in terms of currents and magnetic moments in the m.k.s. system. The concept of magnetic pole is introduced in the discussion of magnetostatics but is used mainly for descriptive purposes. A description of the measurement of susceptibility is included. Chapter 2 deals with the general atomic theory of magnetism and Chapter 3 extends this to ferromagnetism, introducing the ideas of exchange energy and of the collective electron theory. Chapter 4 treats the basic theories of antiferromagnetism and ferrimagnetism in order to explain the properties of ferrites. Chapter 5 is devoted to essentially single-crystal properties of magnetic materials such as anisotropy and magnetostriction. Domains and their influence on the technical magnetization of real materials are discussed in Chapter 6 and these concepts are used in Chapter 7 to account for dynamic behaviour in magnetic materials. This includes a treatment of the basic theory of spin resonance. Chapters 8 and 9 are concerned with real magnetic materials and their properties, which are accounted for in terms of the theory dealt with in the earlier chapters.

In writing a book one incurs great indebtedness to a wide range of people, particularly to the authors of earlier books on the subject, which are quoted in the references. However, I would particularly like to thank Dr R. R. Birss for his careful reading and helpful criticism of the manuscript, Dr S. K. Dey for his assistance with calculations and drawings, L. S. Anderson for his editing of the manuscript, Miss P. J. Carter who typed it and Mrs. M. Rathbone who did the drawings.

J. C. A.

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CHAPTER 1

General Properties

Introduction

Since the first attempts to put its study on a quantitative basis magnetism has been bedevilled by difficulties with units. This has partly arisen through use of the concept of magnetic pole as a discrete entity having positive or negative magnetism, like positive and negative electrostatic charge, and the linking of this idea with the obvious magnetic effects of electric current. Further complications arise in systems in which both electrostatic and electromagnetic units are involved, such as those described by Maxwell's equations. It is in quantitative magnetics that the rationalized m.k.s. system of units shows its greatest value and it is adopted in this book. However, it must be said that when magnetism crosses the frontiers of atomic physics, as must happen in discussing, *inter alia*, theories of magnetic ordering, the m.k.s. units often look unfamiliar. When this is the case attention will be drawn to it. Likewise, despite the strictures on the concept of free magnetic pole given above, there are occasions when it may be used to give a physical picture of what is happening, for example in the discussion of domain walls. Furthermore, the concept of a magnetic dipole *has* reality, and dipole moment is a useful fundamental unit in magnetism. With these caveats in mind we proceed to define the basic magnetic units in the rationalized m.k.s. system.

Magnetic Units

Since magnetic pole cannot be isolated in the same way as can electric charge, the magnetic equivalent to Coulomb's law of force between electric charges cannot easily be experimentally verified. For example, if the repulsive forces between two permanent magnets are used, the exact locations of their north and south 'poles' are hard to define and, in addition to forces of repulsion or attraction, there will generally be a torque on each magnet due to the other. If, however, two long, straight wires are placed parallel to each other and currents are passed through them, a force of attraction arises between them when the currents are in the same direction and repulsion when they are in opposite directions. This force is measurable and, providing that the wires are straight and parallel, it is everywhere in the plane of the wires and perpendicular to their length. These forces arise from the magnetic effects of the currents, and we say that the current sets up a magnetic flux, around the wire carrying it, whose density is **B**. Magnetic flux density, or magnetic induction, **B**, is a vector quantity and lines of magnetic flux density may be drawn, showing the direction of **B** at any point, in the same way as lines of electric field are drawn. The flux density around a wire is proportional to the strength of the current in it and, in the case of the parallel wires discussed above, the force between them is accounted for by taking one of the wires to experience a force due to the flux density set up by the other. This force has a magnitude given by $F = B\,i\,l$, where l is the length of wire carrying a current i. Unit flux density will exist where one metre of wire carrying a current of one ampere experiences a force of one newton. Thus the units of B are newtons per amp-metre; the units are in fact called webers per metre², for reasons which will be seen later.

In the most general case the force exerted on an element of wire **ds** carrying a current i at a place where the flux density is **B** is given by the vector product

$$\mathbf{dF} = i(\mathbf{ds} \times \mathbf{B}) \tag{1.1}$$

which expresses the spatial relations between the directions of **B**, of current i (which is the direction of the element of wire **ds**) and the force **F**, (well known as Fleming's rule for motors) in which the force is perpendicular to both current and flux.

So far no mention has been made of the medium in which the flux density exists; experiment shows that the surrounding medium influences the flux density set up in the vicinity of a current-carrying wire. Equation (1.1), however, suggests that the force is independent of the medium in which it exists, and so the properties of the medium must enter in determining the magnitude of **B** arising from a given current i. To allow for this we introduce the idea of magnetic field **H** defined by the equation

$$\mathbf{B} = \mu\mathbf{H} \tag{1.2}$$

where μ is the property of the medium called its permeability. In free space μ has a fixed value μ_0 and experiments are consistent with the assumption that the force on an element $\mathbf{ds}_1$ carrying a current i_1, placed a distance $\mathbf{r}$ from an element $\mathbf{ds}_2$ carrying a current i_2 is given by

$$\mathbf{dF}_1 = \frac{\mu_0 i_1 i_2}{4\pi r^3}\,\mathbf{ds}_1 \times (\mathbf{ds}_2 \times \mathbf{r}) \tag{1.3}$$

$\mathbf{r}$ is a vector joining the two elements, defined as positive when directed from ds_2 to ds_1. This means that the force is positive, i.e. attractive, when the currents i_1 and i_2 are in the same direction in accordance with experiment. Comparing this with equation (1.1) we see that the flux density has been replaced by $(\mu_0/4\pi r^3)i_2(\mathbf{ds}_2 \times \mathbf{r})$. Using equation (1.2) we may say that the magnetic field due to a current i in an element $\mathbf{ds}$ at a distance $\mathbf{r}$ from it is given by

$$d\mathbf{H} = \frac{d\mathbf{B}}{\mu_0} = \frac{i}{4\pi r^3}\,(\mathbf{ds} \times \mathbf{r}) \tag{1.4}$$

It will be seen that dH falls off inversely with the square of distance as does the electric field due to a point charge.

This equation is, perhaps, best illustrated by considering the case of the field at the centre of a plane circular coil of one turn. Here r is just the radius of the coil and ds, an element of the wire, will everywhere be perpendicular to r, so that the vector product simply becomes $r.ds$. Integrating ds round the coil simply gives $2\pi r$ so that the field at the centre is given by

$$H = \frac{i}{4\pi r^3}.2\pi r.r = \frac{i}{2r} \tag{1.5}$$

Clearly the units of H are amperes per metre and unit field will exist at the centre of a coil of diameter 1 metre carrying a current of 1 amp and will have a direction perpendicular to the plane of the coil.

Returning to equation (1.2), for free space

$$\mathbf{B} = \mu_0 \mathbf{H} \quad \text{or} \quad \frac{\mathbf{B}}{\mathbf{H}} = \mu_0$$

from which μ_0 has the units

$$\frac{\text{newton}}{\text{amp-metre}} \cdot \frac{\text{metre}}{\text{amp}} = \text{newton/amp}^2.$$

This is equivalent to farads/metre and the latter unit is usually used for permeability. The permeability of free space has the value, in the m.k.s. system, of $\mu_0 = 4\pi \times 10^{-7}$ farads/metre.

Returning to the plane coil, the field at a point, *P*, on its axis a distance *x* from the centre of the coil may also be calculated using equation (1.4). Referring to Fig. 1.1 if the coil is of *n* turns and radius *a* and the distance from an element *ds* of the coil to P is *r* then

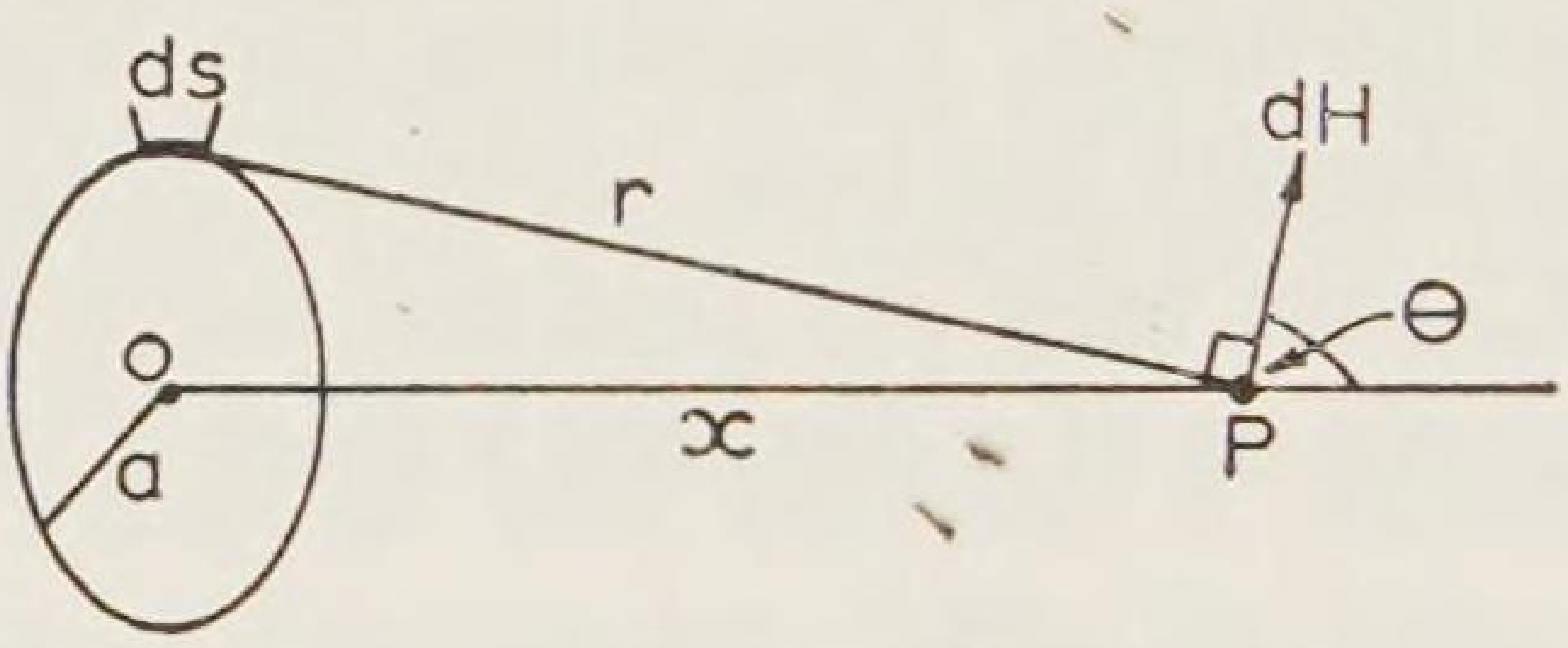


FIG. 1.1

$dH = (ni/4\pi r^3) ds . r$, since *r* and *ds* will always be perpendicular so long as P lies on the axis, *dH* will have a direction perpendicular to *r*, making an angle θ with the axis and so will have components $dH \cos\theta$ along the axis and $dH \sin\theta$ perpendicular to it. Summing round the coil will mean that the radial components sum to zero whilst the axial components add.

Thus

$$H = ni \int \frac{ds}{4\pi r^2} \cos\theta$$

$$= \frac{2\pi nia \cos\theta}{4\pi r^2}$$

But $\qquad \cos\theta = \dfrac{a}{\sqrt{(x^2+a^2)}} \quad \text{and} \quad r^2 = (x^2+a^2)$

Therefore
$$H = \frac{1}{2}\frac{nia^2}{(x^2+a^2)^{3/2}} \tag{1.6}$$

This may be applied to finding the field on the axis of a long solenoid of N turns per metre, carrying a current i. An element of its length dx can be treated as a plane coil of Ndx turns carrying a current i amps. The field on the axis due to such an element will be given by equation (1.6), i.e.

$$dH = \tfrac{1}{2}Nia^2 \frac{dx}{(x^2+a^2)^{3/2}}$$

To sum over the length of the solenoid it is most convenient to change to the variable ϕ shown in Fig. 1.2, from which

$$\frac{a}{(x^2+a^2)^{1/2}} = \sin\phi$$

and
$$dx = -\frac{rd\phi}{\sin\phi}$$

(note increase of x means decrease of ϕ).

Thus
$$H = \int_{\phi_2}^{\phi_1} -\tfrac{1}{2}Ni\sin\phi\, d\phi$$

i.e.
$$H = \tfrac{1}{2}Ni(\cos\phi_1 - \cos\phi_2). \tag{1.7}$$

For an infinite solenoid $\phi_1 = 0$ and $\phi_2 = \pi$, so that

$$H = Ni \tag{1.8}$$

Equation (1.8) may also be used to define unit magnetic field simply as one-Nth of that existing on the axis of an infinite solenoid having N turns per metre and carrying a current of one ampere. Note that this field is independent of the radius of the solenoid.

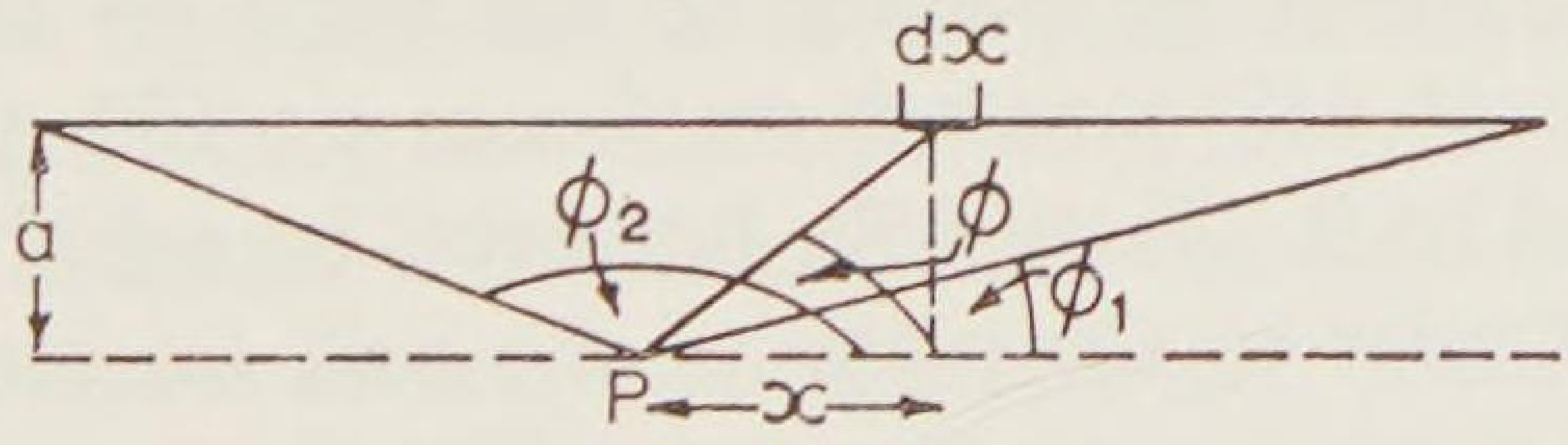


FIG. 1.2

Magnetic Moment

If a plane coil, carrying a current *i*, is placed in a region of uniform magnetic flux, *B*, it will experience a torque whose magnitude depends on the area of the coil, the current, and the component of *B* in the plane of the coil. This may be seen by reference to Fig. 1.3. We consider the coil to be divided up into strips, parallel to the *x*-axis, such as ACDE. The *y*-axis is taken perpendicular to the direction of $B \sin \theta$, which is in the plane of the coil. The flux density *B*, itself, comes out of the paper at an angle θ to the normal and its other component is $B \cos \theta$ perpendicular to the plane of the coil. The elements of coil AE and DC are assumed to be connected by wires CA and ED which carry the current *i*. The whole coil may be made up from such strips and the wires such as CA and ED may then be eliminated, since they carry equal and opposite current, leaving only the coil carrying the current *i*.

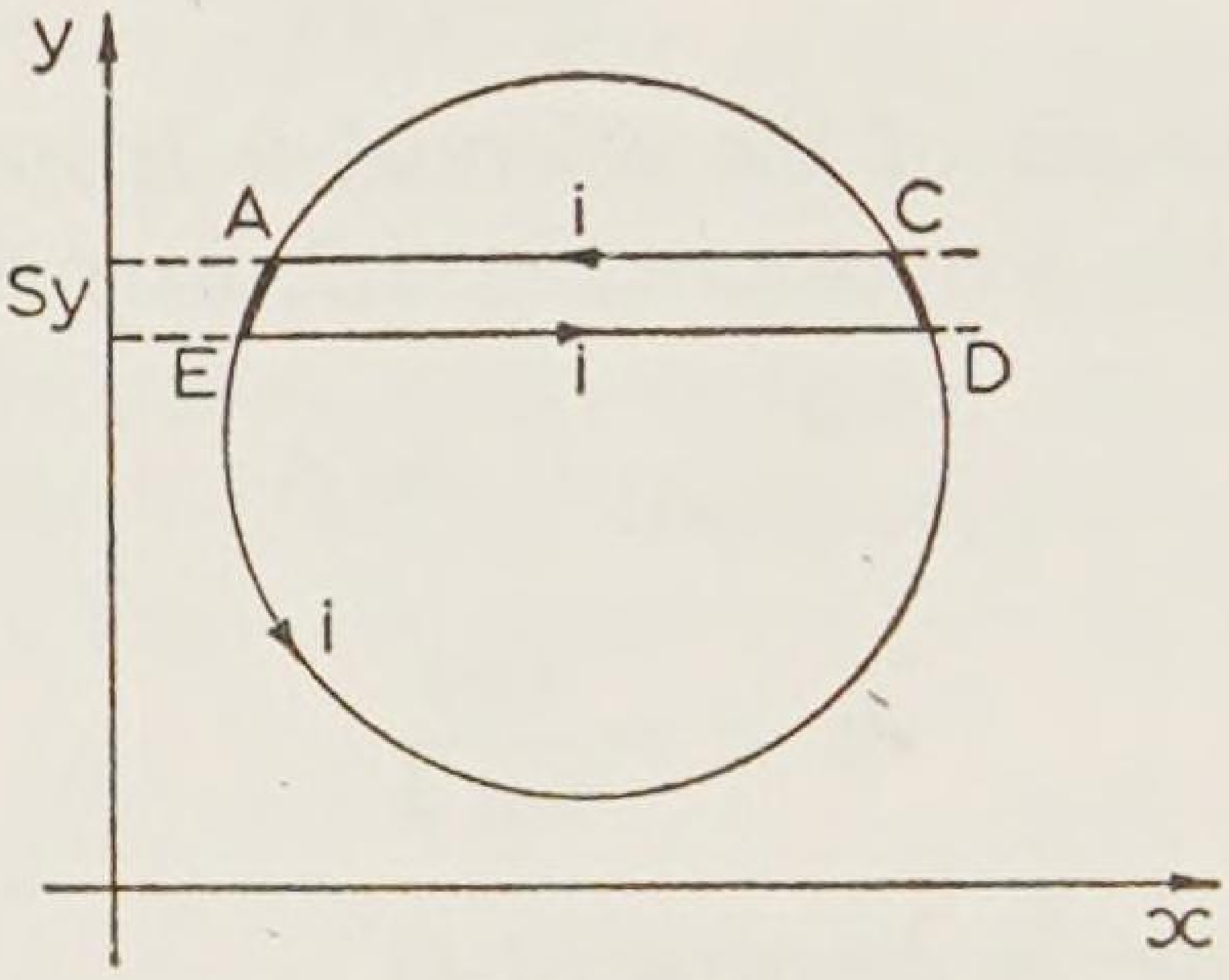


FIG. 1.3

Using equation (1.1) there will be a force on each section of the strip due to $B \cos \theta$ and these forces will be in the plane of the coil, each perpendicular to one of the wires AE, DC, CA and ED. They will be in equilibrium, producing no net force on the strips. The $B \sin \theta$ component will generate no forces on ED and CA, since they are parallel to it. There will, however, be equal and opposite forces, perpendicular to the plane of the coil, on each of AE and DC, producing a torque on the strip. The forces, and hence the torque, will be proportional to the projection of AE and DC on the *y*-axis, these being the elements *ds* (in this case δy) perpendicular to

$B \sin \theta$ in accordance with equation (1.1). Thus the torque on the strip will be given by one of the forces times the distance between them, i.e. $i\delta y B \sin \theta . \mathrm{ED}$, and will tend to turn the strip about the y-axis. Now $\delta y . \mathrm{ED}$ is the area of the strip and can be represented by a vector **dS** having a magnitude equal to the area and direction normal to the strip. The torque can then be written

$$\mathbf{L} = i\mathbf{dS} \times \mathbf{B} \tag{1.9}$$

since the vectors **dS** and **B** have the angle θ between them.

Using equation (1.2), assuming the coil to be *in vacuo*,

$$\mathbf{L} = i\mu_0\mathbf{dS} \times \mathbf{H} = \mathbf{m} \times \mathbf{H} \tag{1.10}$$

where $\mathbf{m} = \mu_0 i\mathbf{dS}$ is defined as the magnetic dipole moment of the strip. The summation of similar strips will give the torque on the whole coil which will thus be

$\mathbf{L} = \mathbf{m} \times \mathbf{H}$, **L** and **m** now referring to the whole coil,

where
$$\mathbf{m} = \mu_0 i \int \mathbf{dS} \tag{1.11}$$

$$= \mu_0 i \times \text{area of coil}$$

and is the dipole moment of the whole coil, being directed perpendicular to its plane.

A permanent bar magnet also experience a torque when placed in a uniform field, which tends to align it with the field. It was this which led to the concept of dipole moment since the magnet was supposed to have free poles of opposite kind, one at each end. The product of the pole strength and distance between them gave the strength of the dipole moment of the magnet which, in turn, described the torque which it would experience in the field. The direction of the dipole moment is coincident with the dipole's length. In the m.k.s. system this is inverted and the strength of the magnet can be described in terms of the maximum torque produced on it in a uniform field of unit strength, this being expressed as its magnetic moment, usually quoted per unit volume of the magnet.

If we assume the magnetic dipole to be equivalent to the coil then, referring to equation (1.11),

$$m = \frac{L}{H \sin \theta}, \text{ and, assuming } \theta = 90° \text{ (for maximum torque)}$$

$$m = \frac{\text{newton-metres}}{\text{amps-(metre)}^{-1}}$$

and $$m \text{ per unit volume} = \frac{\text{newton-metre}^2}{\text{amp-metre}^3}$$

$$= \frac{\text{newton}}{\text{amp-metre}}$$

but this is precisely the unit of flux density B, called weber/metre^2. Thus magnetic moment per unit volume has the same dimensions as flux density. Therefore characterizing a magnet by the maximum torque produced on it in a uniform, unit field is simply another way of expressing the flux density at its end. There is, however, one factor missing in the above use of the equivalence of a coil and a dipole. We have seen that the force vector on the current-carrying coil is **B** and its value is linked to **H** through the permeability of the medium in which it is immersed. Conversely, in the bar magnet (magnetic dipole) experiment it is field **H** which is used in defining its magnetic moment. The immersion of the whole system in a medium of permeability μ means that $B = \mu H$ instead of $\mu_0 H$ and this must be allowed for. It is convenient to write $\mu = \mu_r \mu_0$ where μ_r is called the 'relative permeability' of the medium. It is a dimensionless number and can be shown to equal the permeability of the medium in the c.g.s. system. The effect of the introduction of the new medium would be that we must reduce **H** by a factor $1/\mu_r$ to maintain the same **B**. Since the torque on the coil would then remain the same, equation (1.11) would have to be modified so that

$$\mathbf{m} = \mu_r \mu_0 i \int d\mathbf{S} \tag{1.12}$$

for a coil. This then provides that, for a fixed field, the torque on a coil and on a magnet having the same magnetic moment would be the same, and they are equivalent.

Magnetization

All media are affected by the presence of a magnetic field in that they are found experimentally to acquire a magnetic moment. The magnitude of this moment, per unit volume, is called the *magnetization* of the medium and is described by a vector **M**. From the arguments in the preceding paragraph it will be seen that **M** has the same units and characteristics as **B** and may be regarded as adding to the value of flux density, $\mathbf{B}_0$, which would have been present if the region occupied by the magnetized medium were replaced by vacuum, provided that the field itself is not modified by the introduction of the magnetizable material. This is true if, for instance, the material is in the form of a toroid. The modification necessary when this is not so is dealt with later in connection with demagnetizing fields. Thus we may write the *total* flux density in the medium as

$$\mathbf{B} = \mathbf{B}_0+\mathbf{M} = \mu_0\mathbf{H}+\mathbf{M} \tag{1.13}$$

Alternatively we may say that the magnetization depends on $\mathbf{B}_0$ and equals B_0 multiplied by some factor, χ, which depends on the properties of the medium, i.e.

$$\mathbf{M} = \chi\mathbf{B}_0 = \chi\mu_0\mathbf{H} \tag{1.14}$$

where χ is called the *magnetic susceptibility* of the medium and is a pure number. Using (1.13) and (1.2)

$$\mathbf{B} = \mu\mathbf{H} = \mu_r\mu_0\mathbf{H} = \mu_0\mathbf{H}+\mathbf{M} = \mu_0\mathbf{H}+\chi\mu_0\mathbf{H}$$

we have

$$\chi = (\mu_r-1) \tag{1.15}$$

Media are generally classified by the sign and magnitude of their susceptibilities. By far the majority of solids are *diamagnetic* for which χ is negative, typically in the region of -10^{-5}. This has the implication that **M** is negative, from equation (1.14), so that a diamagnetic medium *reduces* the flux density when it is placed in a field.

When χ is small and positive, typically $\sim 10^{-3}$ at room temperature, the substance is called paramagnetic and is characterized by χ being inversely proportional to temperature. *Ferromagnetic* materials have large, positive values of χ and also show spontaneous magnetization so that **B** is finite in the absence of field and is equal to **M**.

Potential Energy

Since torque is rate of change of potential energy with angle, we may write the potential energy of a dipole in a uniform field as

$$W = -\int \mathbf{L}\,d\theta = -\int (\mathbf{m}\times\mathbf{H})\,d\theta = -\mathbf{m}.\mathbf{H}, \qquad (1.16)$$

since there are no translational forces on the dipole in a uniform field. If the dipole is replaced by a current loop immersed in a magnetic medium, the P.E. of an elementary strip $\mathbf{dS}$ will be given by

$$dW = -\mathbf{m}.\mathbf{H} = -i\mu_r\mu_0\mathbf{dS}.\mathbf{H}$$

so that the total P.E. will be the integral over the surface area of the coil

$$W = -\int i\mu_r\mu_0\mathbf{dS}.\mathbf{H}$$

$$= -\int i\mathbf{dS}.\mathbf{B} = -i\int \mathbf{B}.\mathbf{dS}$$

But $\int \mathbf{B}.\mathbf{dS}$ is simply the total flux normal to the area of the coil, i.e. the total normal component of flux, Φ_N, threading the coil; thus

$$W = -i\Phi_N \qquad (1.17)$$

In the m.k.s. system energy W is in joules so that Φ_N is joules/amp and this is the unit named the weber. It follows that the units of B must be webers/metre^2, and using joules = newton-metres this clearly reduces to the units given earlier.

Magnetostatics

The name magnetostatics is given to the study of the relationships in magnetism comparable to those of electrostatics in electricity. The latter are concerned with the calculation of static field and charge distributions in a dielectric medium. In magnetostatics it is convenient to define a magnetic pole strength, q, analogous to electric charge, such that the force between two point poles q_1 and q_2 is given by

$$\mathbf{F} = \frac{q_1q_2}{4\pi\mu r^2}\,\mathbf{a} \text{ newtons} \qquad (1.18)$$

where **a** is a unit vector pointing from one pole directly away from the other and pole strength is expressed in webers.

A pole is assumed to set up a magnetic field which is defined, at a point, P, a vector distance **r** from a pole of strength q, as the force on unit pole placed at P. Thus field is given, from equation (1.18), as

$$\mathbf{H} = \frac{q}{4\pi\mu r^2}.\mathbf{a} \tag{1.19}$$

Magnetic Potential. It is also convenient to define a scalar magnetic potential ϕ, in terms of the work done in moving a unit pole between two points, A and B, in a medium in which a magnetic field exists, i.e.

$$\phi_A - \phi_B = -\int_B^A \mathbf{H}.\mathbf{dl} \tag{1.20}$$

This may be written as

$$d\phi = -\mathbf{H}.\mathbf{dl}$$

where the two points considered are an infinitesimal distance, **dl**, apart. Writing the distance in terms of its components

$$d\phi = -(H_x\,dx + H_y\,dy + H_z\,dz) \tag{1.21}$$

Now ϕ is a function of x, y and z and its total derivative may be written as

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \tag{1.22}$$

Comparing with equation (1.21) we have

$$\mathbf{H} = -\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

where **i**, **j** and **k** are unit vectors, which, in vector notation, is simply

$$\mathbf{H} = -\text{grad}\,\phi = -\nabla\phi \tag{1.23}$$

Magnetic shell. Referring back to equation (1.12) we see that the magnetic moment due to a single-turn current-carrying coil is of magnitude

$$m = \mu i \times \text{area}$$

and has the direction of a vector **m** perpendicular to the plane of the coil. Suppose we replaced the coil by a thin sheet of magnetic material, magnetized everywhere perpendicular to its surface, such that the sheet had precisely the same total magnetic moment. It would have the same area as the coil if we made its *moment per unit area*, σ, equal to μi. Thus we may replace a current carrying coil by a magnetic shell, having the same configuration, of strength

$$\begin{aligned}\sigma &= \mu i \text{ moment/unit area} \qquad (1.24)\\ &= \mu i \text{ newtons per amp.}\end{aligned}$$

Referring to equation (1.10) the torque on a strip of coil of area dS is given by

$$\mathbf{L} = \mathbf{m} \times \mathbf{H}$$

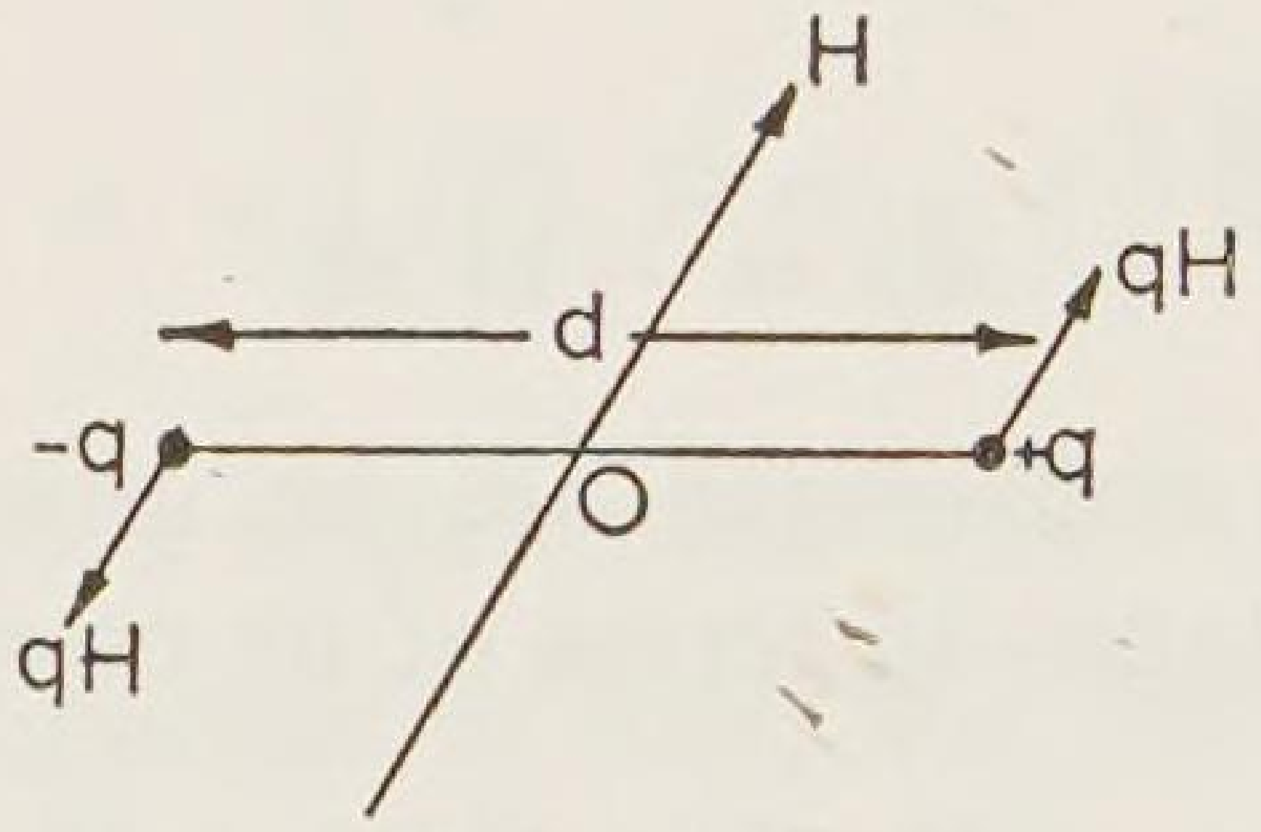


FIG. 1.4

Now consider a magnetic dipole composed of two poles $+q$ and $-q$ separated by a distance d, as illustrated in Fig. 1.4. The force on each pole, from the definition of magnetic field, will be qH, and the torque on the dipole will be given by

$$\begin{aligned}L &= 2.qH \sin \theta . \frac{d}{2}\\ &= qdH \sin \theta .\end{aligned}$$

If d is treated as a vector along the line joining the poles, this may be written as

$$\mathbf{L} = q\mathbf{d} \times \mathbf{H}$$

from which we see that $q\mathbf{d}$ may be identified with magnetic moment, **m**, as already defined.

Applying this to the magnetic shell, if it is visualized as having a thickness, d, the magnetic moment, σ, per unit area will be given by

qd where $+q$ is the pole strength per unit area over one face and $-q$ is the pole strength per unit area over the other face.

The idea of the magnetic shell thus provides a link between the concepts of the m.k.s. system, based on fields and forces due to current, and the older, classical picture of the free magnetic pole.

Field and potential due to a dipole. Referring to equation (1.19) and using the definition of potential given in equation (1.20), the potential at a distance r from a pole $+q$ will be given by

$$\phi_r = -\int_{\infty}^{r} H.dr = -\int_{\infty}^{r} \frac{q}{4\pi\mu r^2}\,dr$$

i.e.

$$\phi_r = \frac{q}{4\pi\mu r} \tag{1.25}$$

where the potential at a point is defined as the work done in bringing a unit pole from infinity to the point.

Referring to Fig. 1.5, the potential at P will be given by

$$\phi_P = \frac{q}{4\pi\mu r_1} - \frac{q}{4\pi\mu r_2} = \frac{q}{\left(r - \frac{d}{2}\cos\theta\right)4\pi\mu} - \frac{q}{\left(r + \frac{d}{2}\cos\theta\right)4\pi\mu}$$

$$= \frac{qd\cos\theta}{\left(r^2 - \frac{d^2}{4}\cos^2\theta\right)4\pi\mu}$$

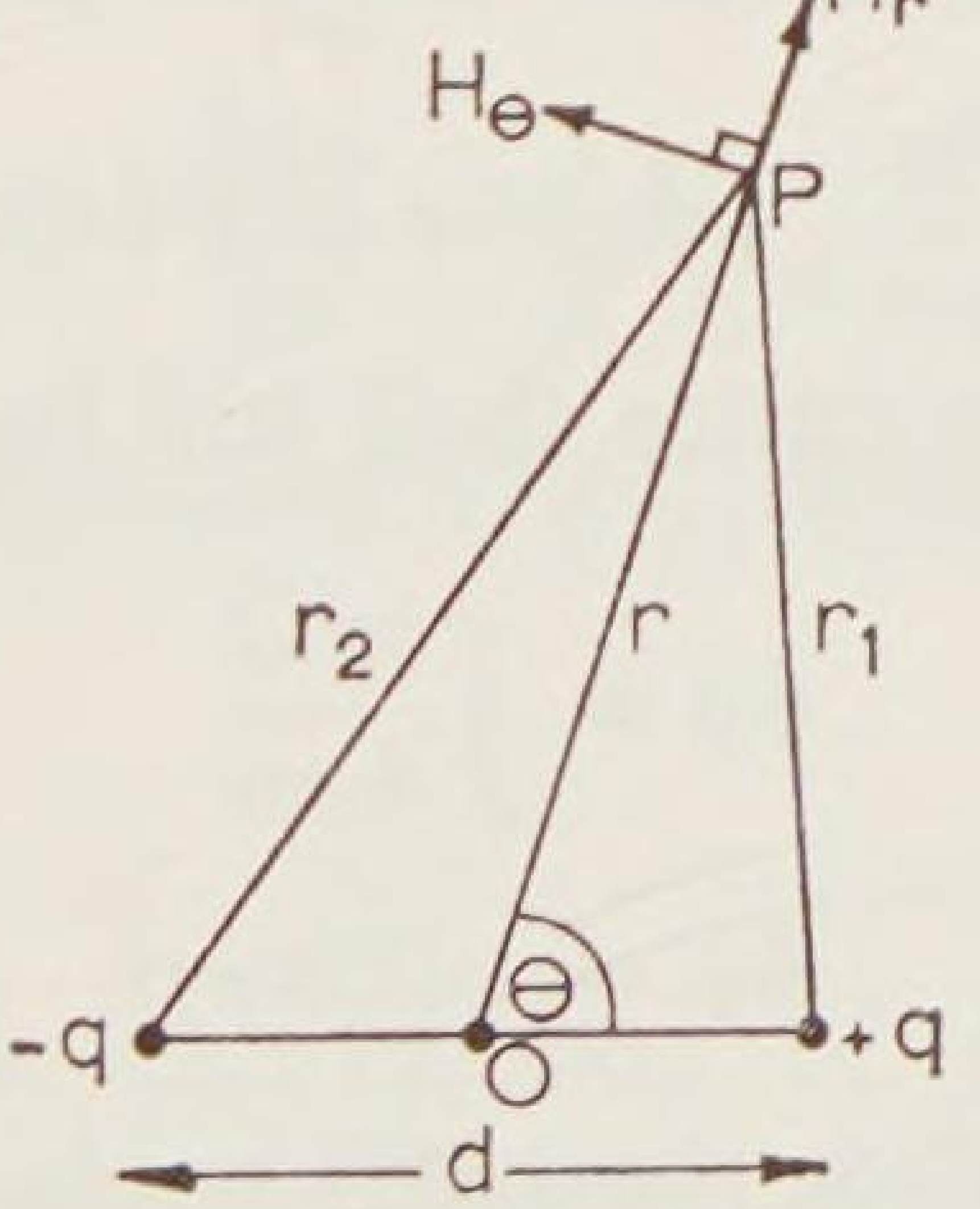


FIG. 1.5

but $d \ll r$, since the dipole is defined as two opposite charges a short distance apart, thus

$$\phi_P = \frac{qd\cos\theta}{4\pi\mu r^2}.$$

This can be written as

$$\phi_P = \frac{\mathbf{m}.\mathbf{r}}{4\pi\mu r^3} \tag{1.26}$$

where **m** is the magnetic moment of the dipole.

By definition, the field components in the direction of, and perpendicular to, r will be given by

$$\left.\begin{aligned} H_r &= -\frac{\partial\phi}{\partial r} = \frac{2m\cos\theta}{4\pi\mu r^3} \\ \text{and}\qquad H_\theta &= -\frac{\partial\phi}{r\partial\theta} = \frac{m\sin\theta}{4\pi\mu r^3} \end{aligned}\right\} \tag{1.27}$$

Field and potential due to a coil. An element of magnetic shell, of area dS and strength σ may be treated as a dipole. Its potential at a point, P, will be given by equation (1.26) as

$$d\phi = \frac{\sigma\, dS\cos\theta}{4\pi\mu r^2}$$

where θ is the angle between the direction r and the direction of the normal to the surface element dS. Now the solid angle, $d\Omega$, subtended by dS at P is given by

$$d\Omega = \frac{dS\cos\theta}{r^2}$$

so that

$$d\phi = \sigma\, d\Omega/4\pi\mu.$$

The potential due to the entire shell will be

$$\phi_P = \int\frac{\sigma d\Omega}{4\pi\mu}$$

i.e.

$$\phi_P = \sigma\Omega/4\pi\mu \tag{1.28}$$

if σ is constant, where Ω is the solid angle subtended by the whole shell.

If the shell is replaced by a coil carrying a current, i, then

$$\phi_P = \frac{i\Omega}{-4\pi} \tag{1.29}$$

Referring to Fig. 1.6, where P is on the axis of the coil of radius a

$$\Omega = \frac{dS\cos\theta}{r^2}.$$

Now $S = \pi y^2 = \pi(r^2 - x^2)$

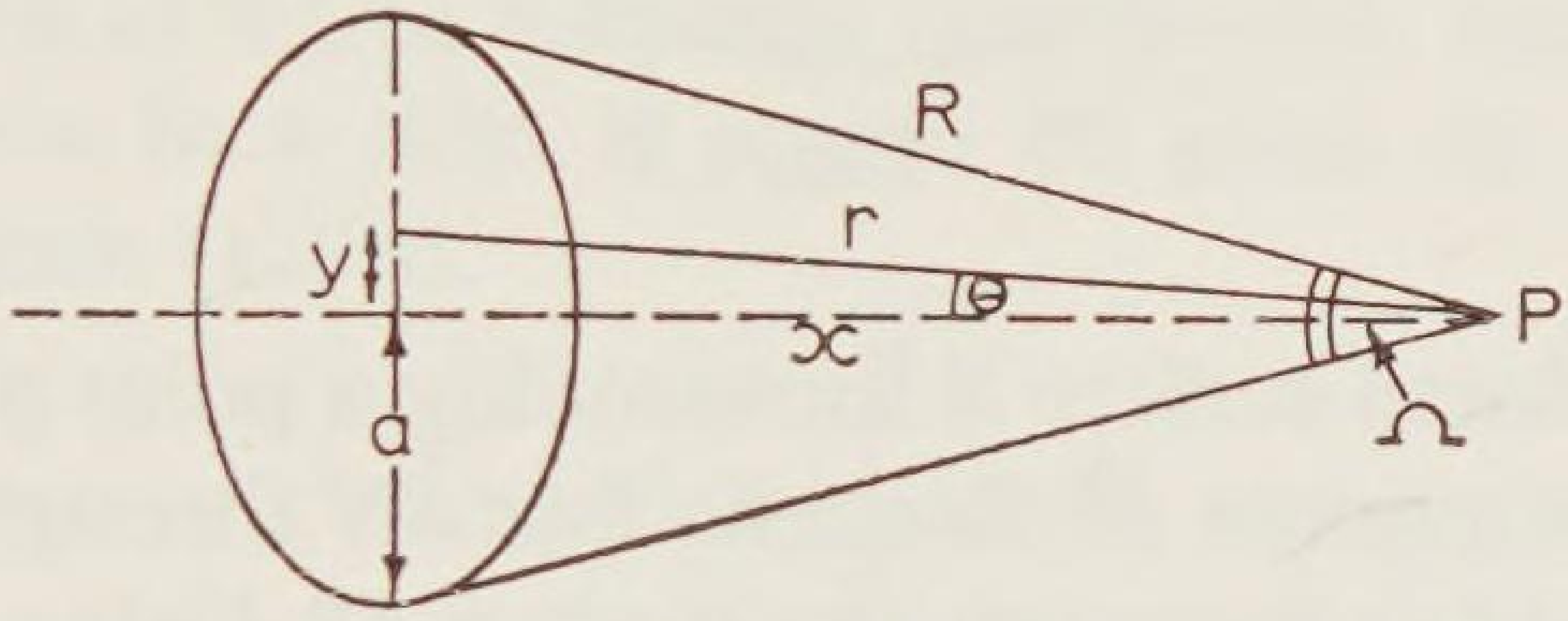


FIG. 1.6

Therefore $dS = 2\pi r dr$ where x is a constant; also $\cos\theta = x/r$. Therefore

$$\Omega = \int_x^R \frac{2\pi x dr}{r^2}$$

$$= 2\pi\left(1 - \frac{x}{R}\right).$$

Thus
$$\phi_P = \frac{1}{2}.i\left(1 - \frac{x}{(x^2+a^2)^{1/2}}\right). \tag{1.30}$$

The field along the axis of the coil will be given by

$$H_x = -\frac{\partial\phi_P}{\partial x} = \frac{1}{2}i.\frac{a^2}{(x^2+a^2)^{3/2}}$$

which is the same as equation (1.6), developed earlier from first principles, with $n = 1$.

Gauss' Law

Having established the equivalence of the magnetostatic and electrostatic concepts we may deduce Gauss' law for a magnetostatic field in the same manner as for an electrostatic field.

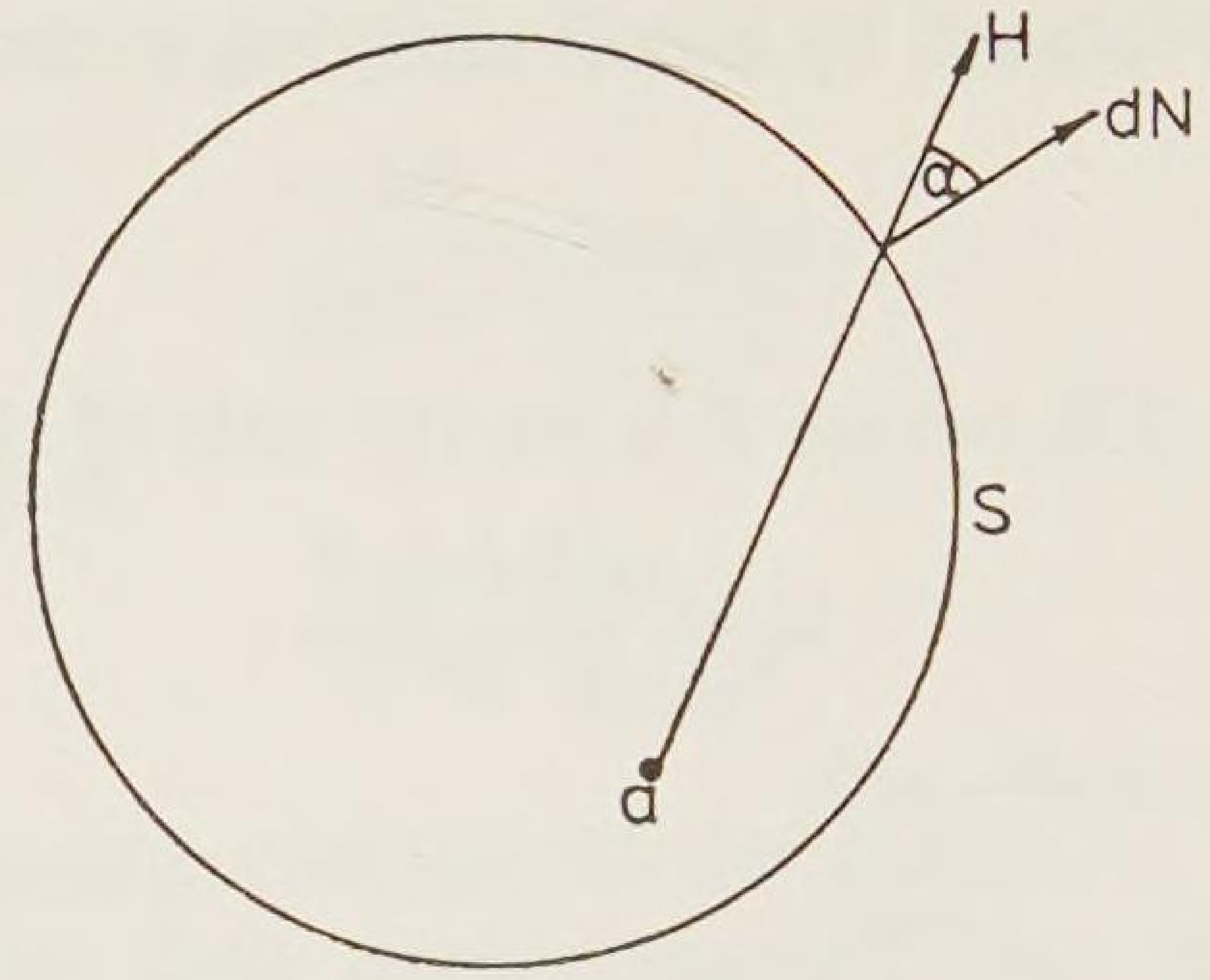


FIG. 1.7

Consider a closed surface S surrounding a point pole q (Fig. 1.7). The normal component of magnetic flux dN through the surface of area dS is defined as the product $\mu H \cos \alpha \, dS$. Using equation (1.19) we have

$$dN = \mu H \cos \alpha \, dS = \frac{q \cos \alpha \, dS}{4\pi r^2}.$$

But the solid angle $d\Omega$ subtended by dS at q is defined by

$$d\Omega = \frac{dS \cos \alpha}{r^2}$$

Therefore $$dN = \frac{q d\Omega}{4\pi}$$

Integrating over the entire surface, the total outward flux is thus

$$N = \frac{q}{4\pi} \int_S d\Omega = \frac{4\pi q}{4\pi}$$

since the solid angle subtended by a closed surface is 4π,

i.e. $$N = q \text{ webers.} \tag{1.31}$$

Writing for N the integral of the normal component of flux density over the surface we have

$$\oint_S B \cos \alpha \, dS = q$$

i.e. $$\oint_S \mathbf{B} . d\mathbf{S} = \Sigma q$$

where more than one pole is contained inside the surface. This is the usual form of Gauss' law for magnetostatics. However, we have seen that free magnetic poles can have no existence. If they occur, they do so in pairs as dipoles having poles of equal and opposite value. Thus the correct statement of the Gauss law is

$$\oint_S \mathbf{B}.d\mathbf{S} = 0 \tag{1.32}$$

If such a thing as a free magnetic pole existed, the work done in moving unit pole in a field **H** would be given by the change in potential (equation (1.20)).

i.e.
$$\phi_{\mathrm{AB}} = -\int_{\mathrm{A}}^{\mathrm{B}} \mathbf{H}.d\mathbf{l} \tag{1.33}$$

By analogy with the electrostatic case the quantity $\int \mathbf{H}.d\mathbf{l}$ is called the *magneto-motive force*, or m.m.f.

Clearly, if the integral were round a closed path, from A back to A, the work done would be zero. If, however, the closed path contained a current the work done would not be zero. Suppose we have a coil, carrying a current i, and consider taking a unit pole in a closed loop encircling the wire. Replacing the coil by its equivalent magnetic shell we consider taking the pole from a point A on one side of the shell to a point B, just opposite A, on the other side. From equation (1.29), the change in potential will be

$$\phi = \frac{i}{4\pi}(\Omega_{\mathrm{A}} - \Omega_{\mathrm{B}})$$

where Ω_{A} and Ω_{B} are the solid angles subtended by the shell at A and B respectively. However, since the shell is infinitely thin, A and B, in effect, coincide. $\Omega_{\mathrm{A}} - \Omega_{\mathrm{B}}$ is then the solid angle corresponding to a closed path, which is 4π. Thus

$$\phi = \mathbf{H}.d\mathbf{l} = \frac{i}{4\pi}.4\pi = i.$$

Thus, for a closed path encircling a current i,

$$\mathbf{H}.d\mathbf{l} = i. \tag{1.34}$$

If a region of distributed current, of density **J**, is considered, then the current through an element of surface $d\mathbf{S}$ is $\mathbf{J}.d\mathbf{S}$ and the total current enclosed by the closed loop gives

$$\oint \mathbf{H}.d\mathbf{l} = \int_S \mathbf{J}.d\mathbf{S} \tag{1.35}$$

By Stokes' theorem the left-hand side may be re-written so that

$$\int \text{curl}\,\mathbf{H}.d\mathbf{S} = \int \mathbf{J}.d\mathbf{S}.$$

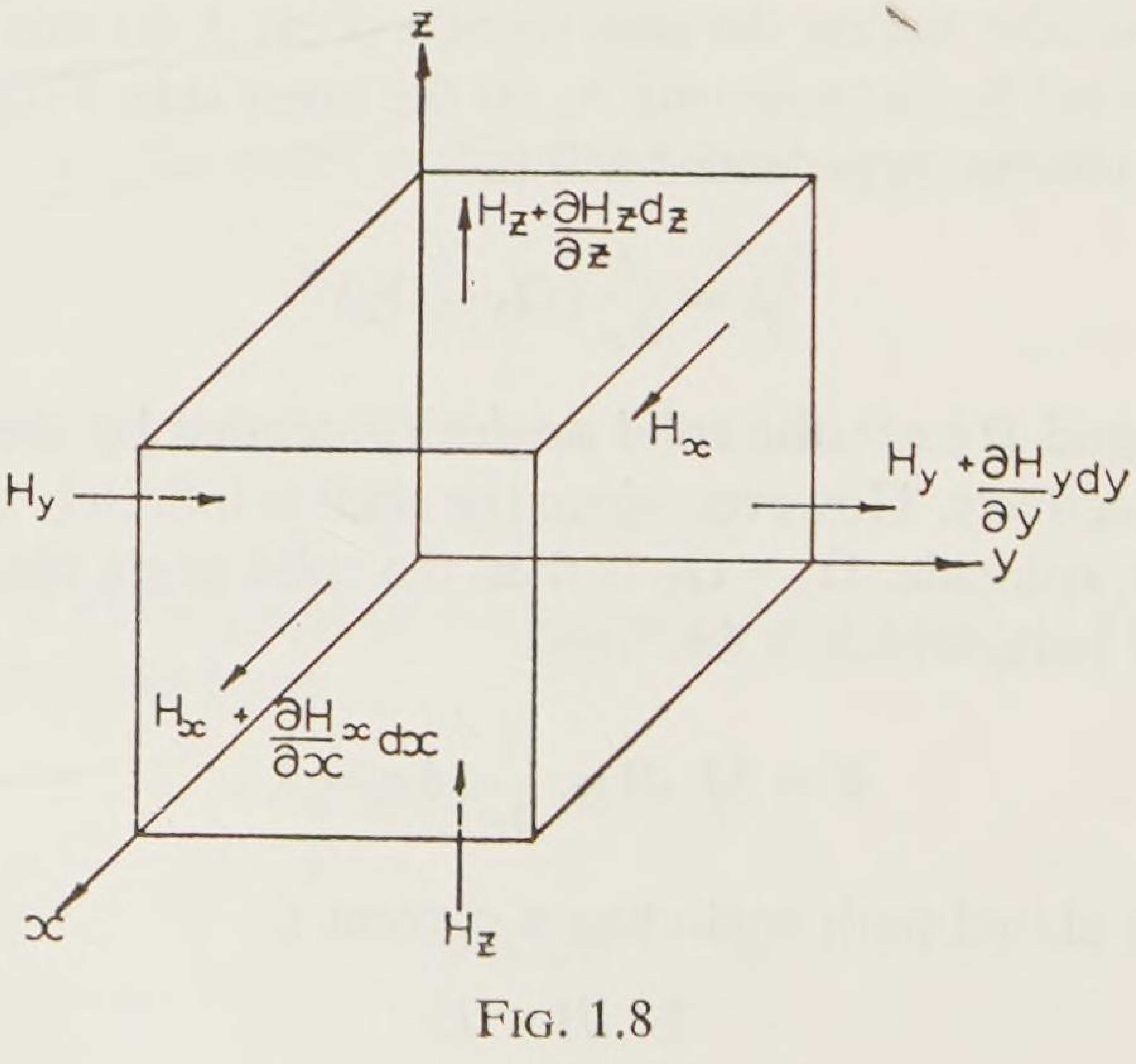
Hence
$$\text{curl}\,\mathbf{H} = \mathbf{J} \tag{1.36}$$

since S may be chosen arbitrarily.
In the absence of a current, of course,

$$\text{curl}\,\mathbf{H} = 0.$$

Laplace's Equation

Consider an elementary volume $dx\,dy\,dz$ as illustrated in Fig. 1.8 in a non-uniform field **H** whose components are H_x, H_y and H_z. The field strength will change between each pair of opposite faces of the



z
$H_z + \frac{\partial H_z}{\partial z} dz$
H_x
H_y
$H_y + \frac{\partial H_y}{\partial y} dy$
y
$H_x + \frac{\partial H_x}{\partial x} dx$
x
H_z

FIG. 1.8

cube so that if the field into one face is, say, H_x, that at the opposite face will be

$$H_x + \frac{\partial H_x}{\partial x}\,dx$$

as shown in the diagram.

The total normal flux emerging from the cube will be given by taking the difference between flux density times area over each pair of faces. Thus the total flux out in the x-direction will be

$$\mu\left(H_x\,dy\,dz + \frac{\partial H_x}{\partial x}\,dy\,dz - H_x\,dy\,dz\right)$$

and similarly for the other directions. Thus the total flux out will be

$$\mu\left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}\right)dx\,dy\,dz$$

Using (1.33) we have

$$\oint_S \mu\mathbf{H}.d\mathbf{S} = \mu\left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}\right)dV = 0 \qquad (1.37)$$

In vector notation

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = \operatorname{div}\mathbf{H} = \nabla.\mathbf{H}$$

and is the divergence of the field. Thus equation (1.34) may be written as

$$\left.\begin{aligned} \mu\nabla.\mathbf{H} &= 0 \\ \text{or} \qquad \nabla.\mathbf{B} &= 0 \end{aligned}\right\} \qquad (1.38)$$

and Gauss' theorem may be restated as Gauss' divergence theorem

$$\oint_S \mathbf{H}.d\mathbf{S} = \int_V \nabla.\mathbf{H}\,dV. \qquad (1.39)$$

Using equation (1.23), $\mathbf{H} = -\nabla\phi$, we obtain, from equation (1.38)

$$\nabla.\nabla\phi = 0$$

i.e.

$$\nabla^2\phi = 0 \qquad (1.40)$$

which is Laplace's equation. The del-squared operator ∇^2 represents the operation

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

This operation may be put in other co-ordinates, and this is sometimes required for particular problems. In cylindrical co-ordinates (ρ, z, ψ)

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\phi}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\psi^2}+\frac{\partial^2\phi}{\partial z^2} \tag{1.41}$$

and in spherical co-ordinates (r, θ, ψ)

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2\phi}{\partial\psi^2}+\frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) \tag{1.42}$$

Demagnetizing Factor

An example of the use of the above magnetostatic equations is given below, which brings out the important idea of demagnetization effects in a magnetizable medium.

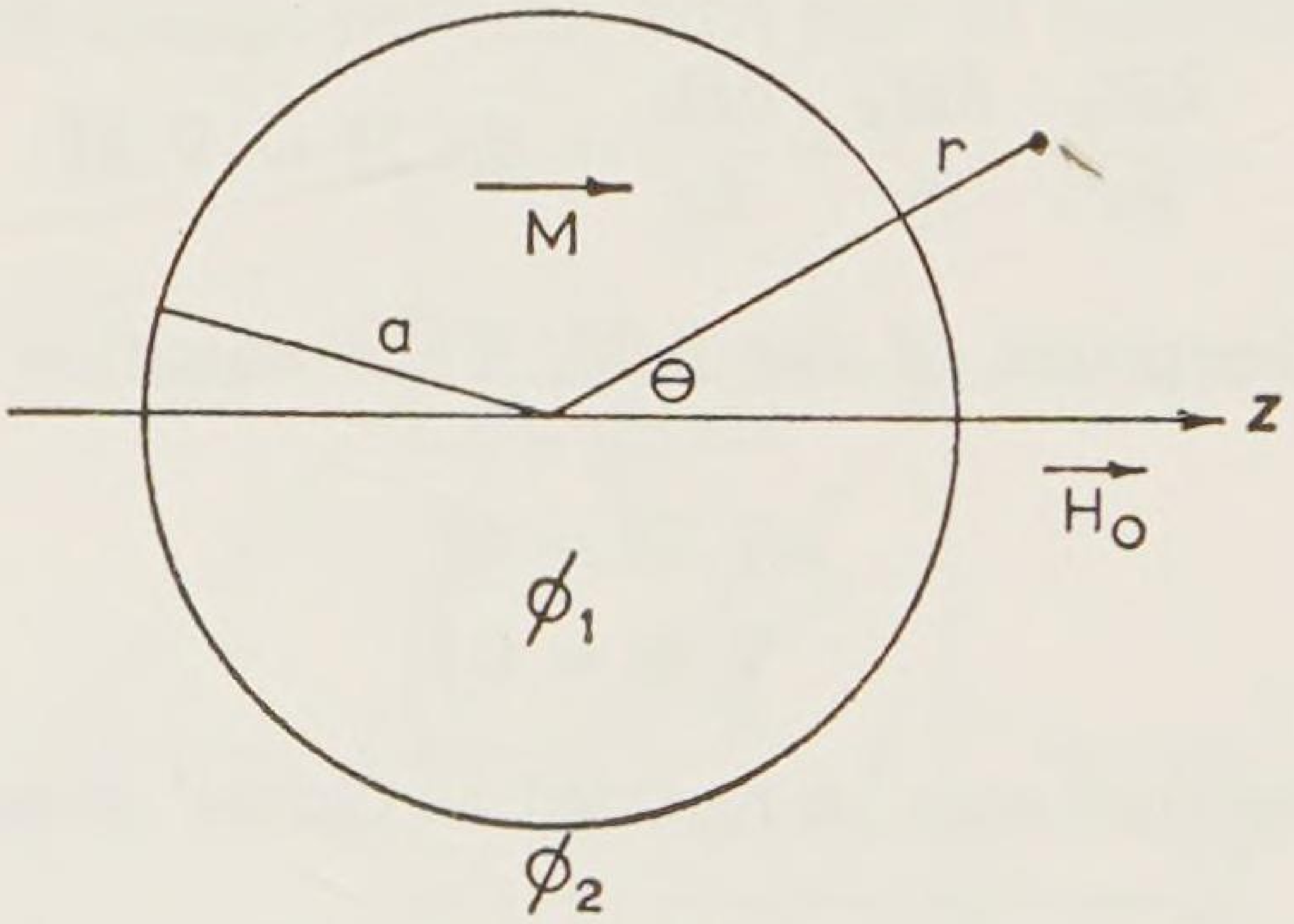


FIG. 1.9

Consider a sphere of magnetizable material having a radius a in a uniform field H_0, as shown in Fig. 1.9, with H_0 directed along the z-axis. The magnetic potential inside the sphere is ϕ_1 and outside it is ϕ_2. We shall make use of the relation of equation (1.40) i.e.

$\nabla^2\phi = 0$. Since the arrangement has spherical symmetry it is convenient to use eqn. (1.42) in spherical co-ordinates, and these may be simplified since H is directed along a polar axis and there will be no variation of field with the azimuthal co-ordinate ψ.
Thus from (1.42)

$$\nabla^2\phi = \frac{r\,\partial^2\phi}{\partial r^2}+\frac{2\partial\phi}{\partial r}+\frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2}+\frac{1}{r\tan\theta}\frac{\partial\phi}{\partial\theta} = 0.$$

Assuming a solution in the form $\phi = R\Theta$ where R is a function of r alone and Θ is a function of θ alone, then

$$rR''\Theta+2R'\Theta+\frac{1}{r}R\Theta''+\frac{1}{r\tan\theta}R\Theta' = 0$$

and
$$\frac{r^2R''}{R}+\frac{2rR'}{R} = -\frac{\Theta''}{\Theta}\,\frac{\Theta'}{\Theta\tan\theta}$$

If the two sides of this equation are to be equal to each other for all r and θ, then both sides can be equal only to a constant. Since the constant can be expressed in any way, let it be $m(m+1)$. Then we have

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-m(m+1)R = 0,$$

and
$$\frac{d^2\Theta}{d\theta^2}+\frac{1}{\tan\theta}\frac{d\Theta}{d\theta}+m(m+1)\Theta =$$

The first of these equations has the solution

$$R = C_1r^m+C_2r^{-(m+1)}.$$

where C_1 and C_2 are constants. The solution of the second equation, however, is more difficult, but can be expressed as a series so that the differential equation is satisfied.

For any integer m, the series is defined by the polynomial

$$P_m(\cos\theta) = \frac{1}{2^m m}\left[\frac{d}{d(\cos\theta)}\right]^m(\cos^2\theta-1)^m.$$

This is known as Legendre's equation and the solutions are called Legendre polynomials of order m. Their forms for the first few values of m are given in Table 1.1.

TABLE 1.1

$P_0(\cos\theta) = 1$
$P_1(\cos\theta) = \cos\theta$
$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$
$P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$
$P_4(\cos\theta) = \frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3)$
$P_5(\cos\theta) = \frac{1}{8}(63\cos^5\theta - 70\cos^3\theta + 15\cos\theta)$

The first two terms of the full solution are

$$\phi = C_1 + C_2 r^{-1} + C_3 r\cos\theta + C_4 r^{-2}\cos\theta$$

In the particular problem being considered the field H_0 is in the z direction and $H_0 = -(\partial\phi/\partial z)$. Since $z = r\cos\theta$ we need clearly take only terms in $\cos\theta$ and $C_1 = C_2 = 0$. When $r \to \infty$ the field is H_0 only, since there will be no effect from magnetization of the sphere at large distances. Thus ϕ_2, which is the potential outside the sphere will be given by

$\phi_2 = C_3 r\cos\theta + C_4 r^{-2}\cos\theta$ and when $r \to \infty$ the second term may be neglected. Using $r\cos\theta = z$ we have $\phi_2 = C_3 z$ and $(\partial\phi/\partial z) = C_3 = -H_0$.

$$\therefore \phi_2 = -H_0 r\cos\theta + C_4 r^{-2}\cos\theta \qquad (r>a) \qquad (1.43)$$

Inside the sphere the term in r^{-2} must be missed out since it becomes infinity at $r = 0$. The field inside the sphere will differ from H_0 due to the magnetization; denoting the internal field by H_1 then, by definition, $H_1 = (-\partial\phi_1/\partial z)$, so that

$$\phi_1 = -H_1 r\cos\theta \qquad (r<a) \qquad (1.44)$$

In equations (1.43) and (1.44) the unknowns are C_4 and H_1 which will be found by inserting the boundary conditions. These are:

(1) potential continuous over the boundary so that $\phi_1 = \phi_2$ at $r = a$ and

$$H_1 = H_0 - C_4 a^{-3} \qquad (1.45)$$

(2) normal components of **B** and tangential components of **H** continuous at boundary.

Inside the sphere the magnetization M is parallel to H_1 in the direction of the z axis so that, in any radial direction r, at an angle θ to the z axis

$$B(r) = M(r)+\mu_0 H(r) = M\cos\theta+\mu_0 H_1\cos\theta.$$

Outside the sphere there is no magnetization so that

$$B(r) = \mu_0 H(r) = -\mu_0\frac{\partial\phi_2}{\partial r} = \mu_0(H_0+2C_4 r^{-3})\cos\theta$$

Equating the expressions at $r = a$ we have

$$M\cos\theta+\mu_0 H_1\cos\theta = \mu_0(H_0+2C_4 a^{-3})\cos\theta. \qquad (1.46)$$

Eliminating C_4 with the aid of (1.45) gives

$$H_1 = H_0-\frac{M}{3\mu_0} \qquad (1.47)$$

The importance of this result is that it illustrates the effect of the magnetization of the sphere on the field inside it. This is that the internal field is reduced, by an amount equal to $M/3\mu_0$, below the value of the applied field. For this reason the component $M/3\mu_0$ is referred to as the *demagnetizing field* of the sphere and its particular value is a consequence of the symmetry of the specimen considered, i.e. a sphere.

In general a magnetized material will set up a demagnetizing field which is proportional to its magnetization. When this field is uniform the total field inside the material is given by

$$H_{\text{int}} = H_{\text{app}}-\frac{N_D M}{\mu_0} \qquad (1.48)$$

where N_D is called the demagnetizing factor for the specimen and depends upon its geometry. The exact calculation of N_D is only possible for certain specific shapes and Bozorth [1] has listed values as shown in Table 1.2 for rods and ellipsoids magnetized parallel to the long axis.

TABLE 1.2

Ratio of length to diameter	Rod	Prolate Ellipsoid	Oblate Ellipsoid
0	1·0 (flat disc)	1·0	1·0
1	0·27	0·33 (sphere)	0·33 (sphere)
2	0·14	0·1735	0·2364
5	0·04	0·0558	0·1248
10	0·0172	0·0203	0·0696
20	0·00617	0·00675	0·0369
50	0·00129	0·00144	0·01472
100	0·00036	0·00043	0·00776
200	$< 10^{-4}$	0·000125	0·0039
500		$< 10^{-4}$	0·001567
1000			0·000784
2000			0·000392

Osborn[2] has carried out calculations for the demagnetizing factor of a general ellipsoid. If the demagnetizing factors along the x, y and z axes are N_x, N_y and N_z, then

$$N_x + N_y + N_z = 1 \tag{1.49}$$

For example, for a sphere all the factors are equal and $N_x = N_y = N_z = \frac{1}{3}$, whilst for a long rod of circular cross-section $N_x = 0$, $N_y = N_z = \frac{1}{2}$, where the long axis of the rod is taken as the x-direction. For an infinite plane $N_x = N_y = 0$ and $N_z = 1$, where the z-axis is perpendicular to the plane.

A general expression can be derived for an ellipsoid of revolution. Taking the long axis as the x-direction, the specimen having a length a in this direction, and with $b = c =$ diameter of the solid, then

$$N_x = \left(\frac{1}{e^2} - 1\right)\left\{\frac{1}{2e}\log\left(\frac{1+e}{1-e}\right) - 1\right\} \tag{1.50}$$

where the eccentricity of the ellipse $e = (1 - b^2/a^2)^{1/2}$. In the y or z directions

$$N_y = N_z = \frac{(1 - N_x)}{2}$$

from equation (1.49) above.

If the ellipsoid is oblate, or disc-shaped, with the diameter $a = b$ where $a \gg c$, the demagnetizing factor in the direction of a diameter is given by

$$N_x = N_y = \frac{1}{2}\left\{\frac{(1-e^2)^{1/2}}{e^3}\sin^{-1} e - \frac{(1-e^2)}{e^2}\right\} \tag{1.51}$$

In this case, the demagnetizing factor perpendicular to the plane of this disc, N_z, will be given by

$$N_z = 1 - 2N_x \tag{1.52}$$

Equivalent Gap

If a toroidol ring of magnetic material is magnetized to saturation by means of a coil wound round it, and the field is then removed, the magnetization will fall to a remanent value $M_r = B_r$. If now a small gap, of length g, were cut in the ring, where $g \ll l$, the mean length of the toroid, a field H_G would exist across the gap and a demagnetizing field H_D would be set up in the specimen, reducing its magnetization to some value, M_D given by $H_D = -N_D M_D/\mu_0$. Neglecting fringing effects, the flux density in the gap and in the specimen will be the same, i.e.

$$B = \mu_0 H_G = \mu_0 H_D + M_D$$

From equation (1.33), since the gap and specimen form a closed loop

$$gH_G + (l-g)H_D = 0$$

and thus, combining these equations, we have

$$M_D . g + \mu_0 H_D l = 0$$

and substituting for H_D gives

$$N_D = g/l.$$

Thus, for a given demagnetizing factor, there is an equivalent gap in a toroidal specimen. This is useful in magnetic circuit calculations.

Magnetostatic Energy

Any system of magnets may be built up of magnetic dipoles, a certain amount of work being involved in the process. Thus such a system will have a potential energy, this being referred to as the magnetostatic energy.

We will take a simple model to simulate a magnetizable medium in which we assume the system to be made up of poles $q_1, q_2 \dots q_i$ and the magnetic potential at the position of each free pole to be $\phi_1, \phi_2, \dots \phi_i$. There are as many positive as negative poles and they are built up by bringing small increments from infinity, the same increment being added to each one, so that, at any instant their values are $\alpha q_1, \alpha q_2, \dots \alpha q_i$, where α varies from 0 to 1 as they build up. If ϕ is the final potential at any one pole q, then the potential will be $\alpha\phi$ when the pole is αq. A single step in the magnetizing process consists of increasing each pole from αq to $(\alpha + d\alpha)q$, i.e. the pole is increased by $q\,d\alpha$. Now, by definition, the work done against the magnetic forces in bringing a unit positive pole from infinity is the potential at that point. Hence the work done in the step described above, for a positive pole, is

$$du = \sum q\,d\alpha . \alpha\phi$$

i.e.

$$du = \alpha\,d\alpha \sum q\phi$$

Thus the total work done in establishing the system will be

$$u = \left[\int_0^1 \alpha\,d\alpha\right] \sum_{i=1}^{n} q_i\phi_i$$

where q_i is the ith pole, which may have either sign.

i.e.

$$u = \frac{1}{2} \sum_{i=1}^{n} q_i\phi_i \tag{1.53}$$

which is its magnetostatic energy. It should be noted that, with equal numbers of positive and negative poles, the total work done corresponds to that involved in separating the pairs of positive and negative poles to form dipoles.

If the length of the jth dipole is d_j its magnetic moment will be

$$m_j = q_j d_j$$

whilst the difference in magnetic potential between its two poles is

$$\phi_+ - \phi_- = -H_{\parallel}\, d_j$$

where $H_\parallel$ is the component of magnetic field parallel to the moment m_j. Then

$$q_j\phi_+ - q_j\phi_- = -\left(\frac{m_j}{d_j}\right)(H_\parallel\, d_j)$$
$$= -\mathbf{m}_j.\mathbf{H}_j$$

so that equation (1.53) becomes

$$u = -\frac{1}{2}\sum_{j=1}^{n}(\mathbf{m}_j.\mathbf{H}_j) \tag{1.54}$$

Now in a magnetizable medium, the magnetization M is precisely the magnetic moment per unit volume. Thus, if we take a unit volume, the sum of the individual dipole moments m_j is simply M. The field, H_j, to which each one is subject is the field due to the whole assembly of dipoles which is, for simplicity, assumed to be uniform throughout the specimen. But this is precisely the demagnetizing field, H_D, when the specimen is of such a shape (e.g. ellipsoidal) that the demagnetizing field is uniform. Thus, we may rewrite equation (1.54) as

$$u = -\tfrac{1}{2}\mathbf{H}.\sum_{j=1}^{n}\mathbf{m}_j$$
$$= -\tfrac{1}{2}\mathbf{M}.\mathbf{H}_D \text{ per unit volume} \tag{1.55}$$

By definition $H_D = N_D/\mu_0 M$, where N_D is the demagnetizing factor, and thus the magnetostatic self-energy of a magnetized material is given by

$$u = \frac{1}{2}\frac{N_D}{\mu_0}M^2 \tag{1.56}$$

when the field and magnetization are parallel inside the specimen. This would be the case, for example, for an ellipsoid magnetized parallel to its long axis.

Although the magnetic self-energy has been calculated here from the magnetic pole model, it is possible also to calculate it using the current model. However, the same answer is obtained (see, for instance, W. Fuller Brown [3]) for the shape-dependent energy.

Diamagnetism

In the simple picture of the atom the electron is taken to move in a closed orbit about the central nucleus. Since it carries a charge, the electron represents current circulating in a closed loop and it has already been seen that this produces a magnetic dipole moment. Incidentally, such a picture also accounts for the non-existence of a free magnetic pole, since only dipoles are produced even on the atomic scale.

The moment of the electron in orbit will be given by equation (1.11)

$$m = \mu_0 \int i\mathbf{dS}$$

where i is the current equivalent to the circulating electron and the integration gives the total orbital area. For the simple case of a circular orbit of radius r in which the electron, of charge $-e$, rotates at a constant angular velocity ω, since the electron makes $\omega/2\pi$ revolutions per second the equivalent current $i = -e\omega/2\pi$. The magnetic moment is then given by

$$m = -\mu_0 \frac{e\omega}{2\pi} \pi r^2 = -\tfrac{1}{2}\mu_0 e\omega r^2.$$

The angular momentum of the electron, G, is given by $m\omega r^2$ so that

$$m = -\frac{\mu_0 e}{2m} G \qquad (1.57)$$

For the more general case of a non-circular orbit we must evaluate the vector **dS** which is equal in magnitude to the area it represents and has the direction of the normal to the surface. Considering an orbit as in Fig. 1.10, let the area **dS** be described by the triangle of which **ds** is the base and **r** the distance from the reference point 0. The area of this triangle is $\frac{1}{2}(ds \sin\theta).r.$ and, if we write this as $\frac{1}{2}(\mathbf{ds}\times\mathbf{r})$ it is a vector of magnitude equal to the area and direction normal to it. Thus the expression for magnetic moment can be written as

$$m = \mu_0 \int \tfrac{1}{2} i(\mathbf{ds}\times\mathbf{r}).$$

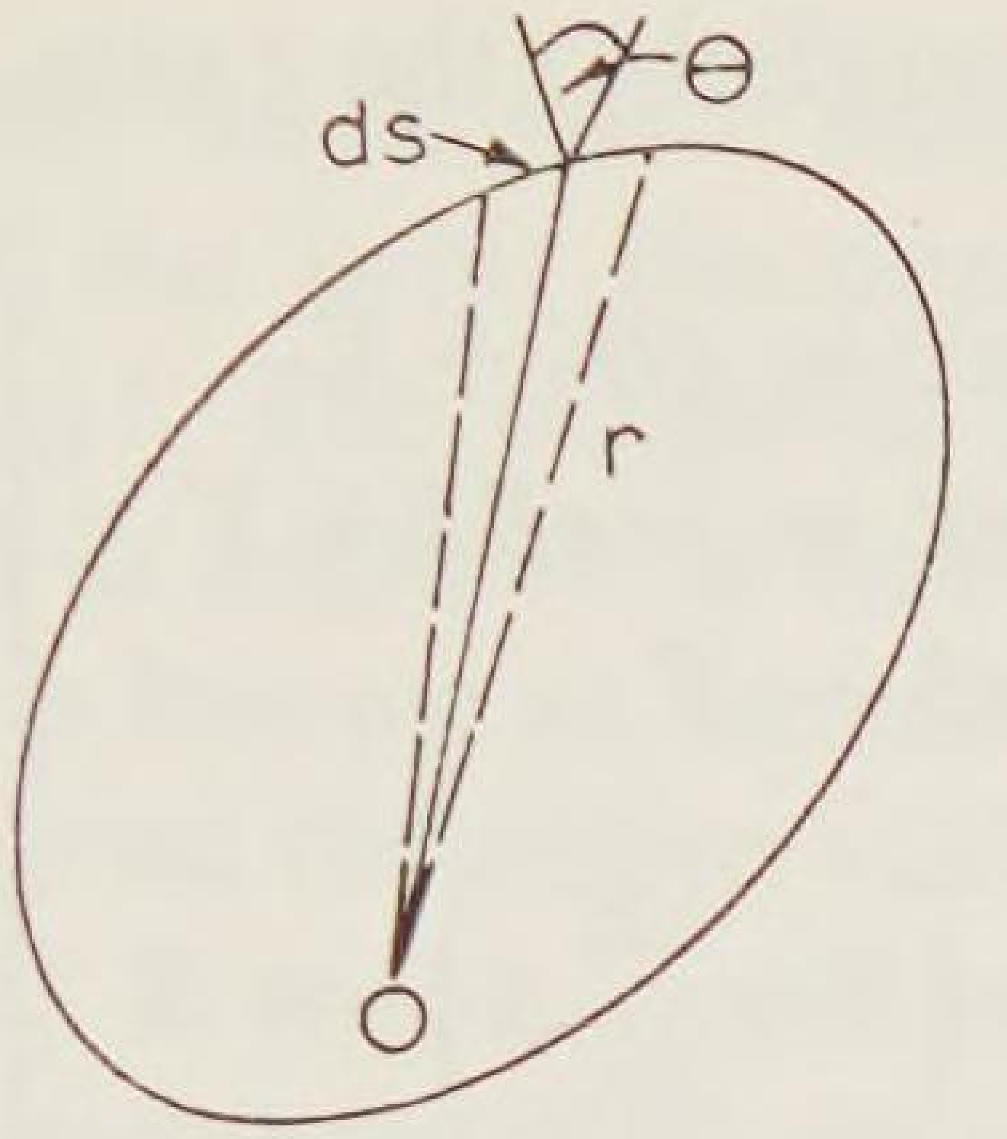


FIG. 1.10

The magnitude of the current will be given by $i = dq/dt$ where q is charge, and if it moves with a linear velocity $\mathbf{v}$ then $i\,\mathbf{ds} = (dq/dt)\mathbf{v}\,dt = dq.\mathbf{v}$; thus

$$m = \tfrac{1}{2}\mu_0 \int dq(\mathbf{v} \times \mathbf{r}).$$

But angular momentum of a particle of mass m is an orbit is, by definition

$$\mathbf{G} = m(\mathbf{v} \times \mathbf{r})$$

and, in the absence of external forces, this is a constant. Thus

$$\mathbf{m} = \mu_0 \frac{\mathbf{G}}{2m} \int dq$$

i.e.
$$\mathbf{m} = -\frac{\mu_0 e\mathbf{G}}{2m} = \gamma\mathbf{G} \tag{1.58}$$

where $-e$ is the total electronic charge and $1/\gamma = -2m/\mu_0 e$ is known as the gyromagnetic ratio.

We now wish to determine the effect, on this magnetic moment, of the application of a magnetic field giving rise to a flux density **B**, acting on the electron, at some angle to the orbit. If we take a central force, X, Y, Z acting on the electron along fixed x, y and z axes the total force on the electron will be the sum of the central force and that due to the magnetic flux. The electron is equivalent to a current equal to charge × velocity and the force on it is given by equation

(1.1). If we take B to be parallel to the z axis, the equations of motion for the electron are

$$\left.\begin{aligned} m\ddot{x} &= eB\dot{y}+X \\ m\ddot{y} &= -eB\dot{x}+Y \\ \text{and} \qquad m\ddot{z} &= Z \end{aligned}\right\}$$

i.e.

$$\left.\begin{aligned} m\left(\ddot{x}-\frac{eB}{m}\dot{y}\right) &= X \\ m\left(\ddot{y}+\frac{eB}{m}\dot{x}\right) &= Y \\ m\ddot{z} &= Z \end{aligned}\right\} \tag{1.59}$$

If now we consider the magnetic field removed but refer the electron motion to axes rotating about the z-axis in a right-handed direction with a constant angular velocity Ω, then the three components of electron velocity can be written as

$$u = \dot{x}-y\Omega, \qquad v = \dot{y}+x\Omega \quad \text{and} \quad w = \dot{z}.$$

The accelerations will be

$$\dot{u}+v\Omega, \qquad \dot{v}+u\Omega, \qquad \dot{w}$$

or, substituting for u, v, w,

$$\ddot{x}-2\dot{y}\Omega-x\Omega^2, \qquad \ddot{y}+2\dot{x}\Omega-y\Omega^2, \quad \ddot{z}$$

which gives the equations of motion

$$\left.\begin{aligned} m(\ddot{x}-2\Omega\dot{y}-\Omega^2 x) &= X \\ m(\ddot{y}+2\Omega\dot{x}-\Omega^2 y) &= Y \\ m\ddot{z} &= Z \end{aligned}\right\} \tag{1.60}$$

We now make the assumption, to be justified later, that terms in Ω^2 are negligible and then compare equations (1.59) and (1.60). From the comparison it can be seen that, if $\Omega = (eB/2m)$, the equations are identical. From this it may be concluded that the effect of the field is the same as that of impressing a rotation, of angular velocity Ω, on the frame of reference. In physical terms this means that the angular momentum, **G**, rotates about the direction of **B** at an angular velocity Ω. Since the angular momentum has a direction normal to the plane of the orbit the normal precesses about the field as shown in Fig. 1.11. The magnetic moment **m** is oppositely directed to **G** (because γ is negative in equation (1.58)) and so also precesses about **B**. This is known as Larmor precession since it was Larmor who first carried out the above analysis. Since $\Omega = (eB/2m)$ and a likely maximum of B is $\sim$2 weber/metre^2, (20,000 gauss), Ω has a value $\sim 1{\cdot}5 \times 10^{11}$ rads/sec. Now an electron in its orbit has a rotation frequency around the frequency of visible light, $\omega_0 \doteqdot 10^{15}$ rads/sec. The acceleration term will be of the order $\omega_0{}^2$ and the velocity term of order $\omega_0 r$, taking, for simplicity, $x = y = r =$ radius of orbit. The term neglected is thus small compared with the other two in equation (1.60).

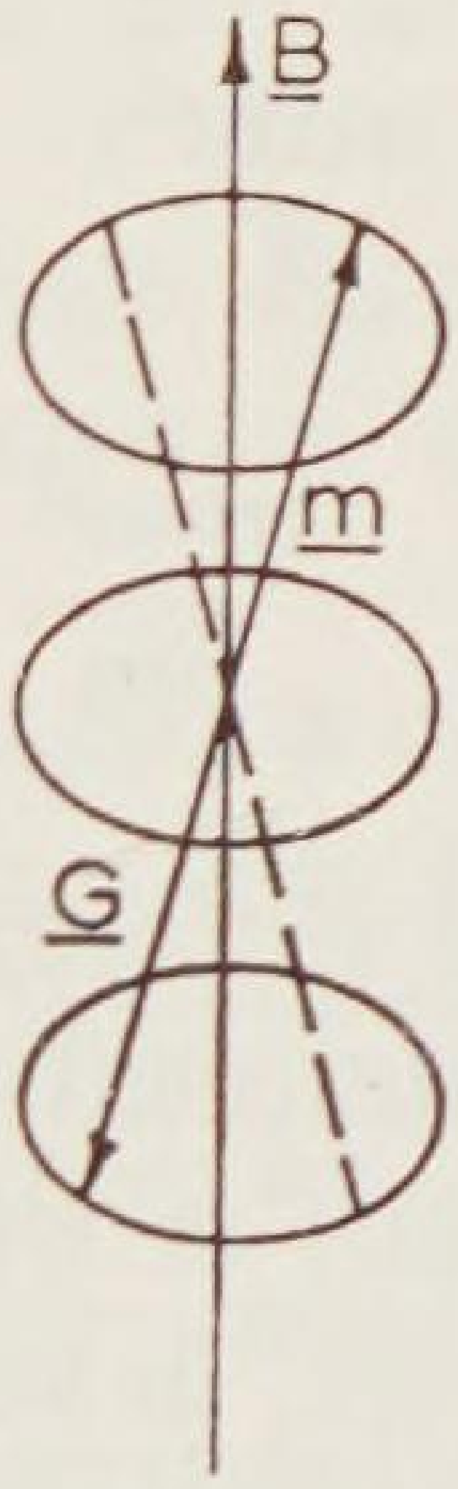


FIG. 1.11

As will be seen from Fig. 1.11, each electron acquires, as a result of precession, a component of angular momentum $\Delta\mathbf{G}$, along the direction of **B**. Now

$$\Delta\mathbf{G} = m(\bar{a})^2\Omega = m(\bar{a})^2\left(\frac{e}{2m}\right)\mathbf{B} \tag{1.61}$$

where $(\bar{a})^2$ is the mean square distance of an electron from an axis, parallel to **B**, through the nucleus. If the mean orbital radius is $\bar{r}_0$ then

$$\bar{r}_0^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2$$

and from the above

$$\bar{a}^2 = \bar{x}^2 + \bar{y}^2 = \tfrac{2}{3}\bar{r}_0^2$$

The magnetic moment induced by the field will be, from equation (1.58)

$$\Delta\mathbf{m} = -\mu_0 \frac{e}{2m} \Delta\mathbf{G}$$

and, for a random collection of orbits,

$$\Delta\mathbf{m} = -\mu_0 \left(\frac{e}{2m}\right)^2 \mathbf{B}.\tfrac{2}{3}m \sum \bar{r}_0^2$$

i.e.

$$\Delta\mathbf{m} = -\mu_0 \frac{e^2}{6m} \sum \bar{r}_0^2 \mathbf{B}.$$

Hence the susceptibility of a sample containing n atoms per unit volume will be

$$\chi = \frac{n\Delta\mathbf{m}}{\mathbf{B}}$$

i.e.

$$\chi = -\frac{n\mu_0 e^2}{6m} \sum \bar{r}_0^2. \tag{1.62}$$

This shows that diamagnetic susceptibility, due to orbital motion, is always negative, as was mentioned earlier. From the fundamental nature of the above treatment it will be realized that all atoms exhibit diamagnetism. However, in some materials the diamagnetic susceptibility may be more than cancelled by a positive paramagnetic component. Since r_0 is virtually temperature independent so also will be the diamagnetic susceptibility. If equation (1.62) is used to evaluate r_0, for a diamagnetic substance, the value comes out to $\sim 10^{-8}$ cm, in good agreement with x-ray data.

Paramagnetism

The proportionality between magnetic moment and angular momentum of the electron was shown in equation (1.58). Orbital motion is not the only contribution to the angular momentum of the electron since it also has spin. The total angular momentum is the vector sum of the spin and orbital components and the magnetic moment is related to this sum. If the total angular momentum of the atom is zero there will be no magnetic moment, but most free atoms have a finite angular momentum and therefore a magnetic dipole moment. However, the free atoms are almost invariably formed into molecules and the chemical bonds between the atoms are usually such that the net magnetic moment of the molecule is just zero. When this is not so and the ions or molecules have a net magnetic moment then they will normally be randomly oriented and the net magnetic moment of the whole collection will be zero. Application of a magnetic field, however, will cause the individual dipoles to try to align with field and so produce a net magnetic moment in the body as a whole. This is paramagnetism.

The experimental observation, first made by Curie and known as the Curie law of paramagnetic susceptibility, is

$$\chi_{\text{para}} = \frac{M}{\mu_0 H} = \frac{\text{const}}{T} \tag{1.63}$$

where T is absolute temperature. Langevin provided a theoretical explanation of this on the basis of classical Boltzmann statistics which is given below.

It is assumed that each atom has a dipole moment m and that the interactions between the atoms can be ignored, so that the only force on each dipole is that due to the applied magnetic field $\mathbf{H}$. The atoms are subject to thermal agitation and Langevin assumed that their energy distribution could be described by classical Maxwell-Boltzmann statistics, so that the number of atoms, dn, having energies between w and $w+dw$ will be given by

$$dn = c\exp\left(\frac{-w}{kT}\right)dw \tag{1.64}$$

where k is Boltzmann's constant, T the absolute temperature and c is a constant. Now, by equation (1.16), the magnetic potential energy of a dipole whose axis makes an angle θ with a field H is

$$w = -mH\cos\theta \tag{1.65}$$

so that

$$dw = mH\sin\theta\, d\theta. \tag{1.66}$$

Combining these equations we have

$$dn = c\exp\left(\frac{mH\cos\theta}{kT}\right)mH\sin\theta\, d\theta \tag{1.67}$$

The total number of atoms, n, is just $\int dn$ over all angles between 0 and π. The total magnetization may be described as $M = n\bar{m}$ where $\bar{m}$ is the average dipole moment per atom in the direction of **H**. The component of each dipole moment in the direction of **H** is $m \cos \theta$ and the sum of these, $\int m \cos \theta \, dn$, will give the total magnetization, thus

$$M = n\bar{m} = m \int \cos \theta \, dn \tag{1.68}$$

i.e.

$$\bar{m} \int_0^\pi dn = m \int_0^\pi \cos \theta \, dn$$

and using (1.67)

$$\frac{\bar{m}}{m} = \frac{\int_0^\pi \cos \theta . c \exp\left(\frac{mH \cos \theta}{kT}\right) . mH \sin \theta \, d\theta}{\int_0^\pi c \exp\left(\frac{mH \cos \theta}{kT}\right) mH \sin \theta \, d\theta}.$$

If we write $a = (mH/kT)$ and $x = \cos \theta$ we have

$$\frac{\bar{m}}{m} = \frac{\int_{-1}^{+1} x \exp(ax) \, dx}{\int_{-1}^{+1} \exp(ax) \, dx} = \coth a - \frac{1}{a} = L(a) \tag{1.69}$$

$L(a)$ is known as the Langevin function and is plotted as a function of a in Fig. 1.12. When a is large the value of $L(a)$ tends to unity, which corresponds to $\bar{m} = m$ and all the dipoles are aligned with the field. However $a = (mH/kT)$ and a typical valve for m is $\sim 10^{-28}$ weber-metre. Thus, assuming $T = 300°K$ and a strong field of 10^7 amps/metre ($\sim$60,000 oersted) a is only of the order of 0·25, from which it may be seen that it is impossible to magnetize the assembly to saturation at room temperature in fields normally available.

More usually a may be treated as small so that the Langevin function approximates to $a/3$ and

$$\frac{\bar{m}}{m} = \frac{mH}{3kT} \tag{1.70}$$

from which the susceptibility is given by

$$\chi = \frac{M}{\mu_0 H} = \frac{n\bar{m}}{\mu_0 H} = \frac{nm^2}{3\mu_0 kT} \tag{1.71}$$

where n is the number of atoms per unit volume. Clearly equation (1.71) is the same as (1.63) where the const. $C = (nm^2/3\mu_0 k)$ and is known as the Curie constant for the particular material. Susceptibility is frequently quoted as a value per unit mass or per gramme-molecule. The mass susceptibility is simply χ/ρ where ρ is density whilst the susceptibility per gramme-molecule is $\chi/\rho M$ where M is the molecular weight in kilogrammes.

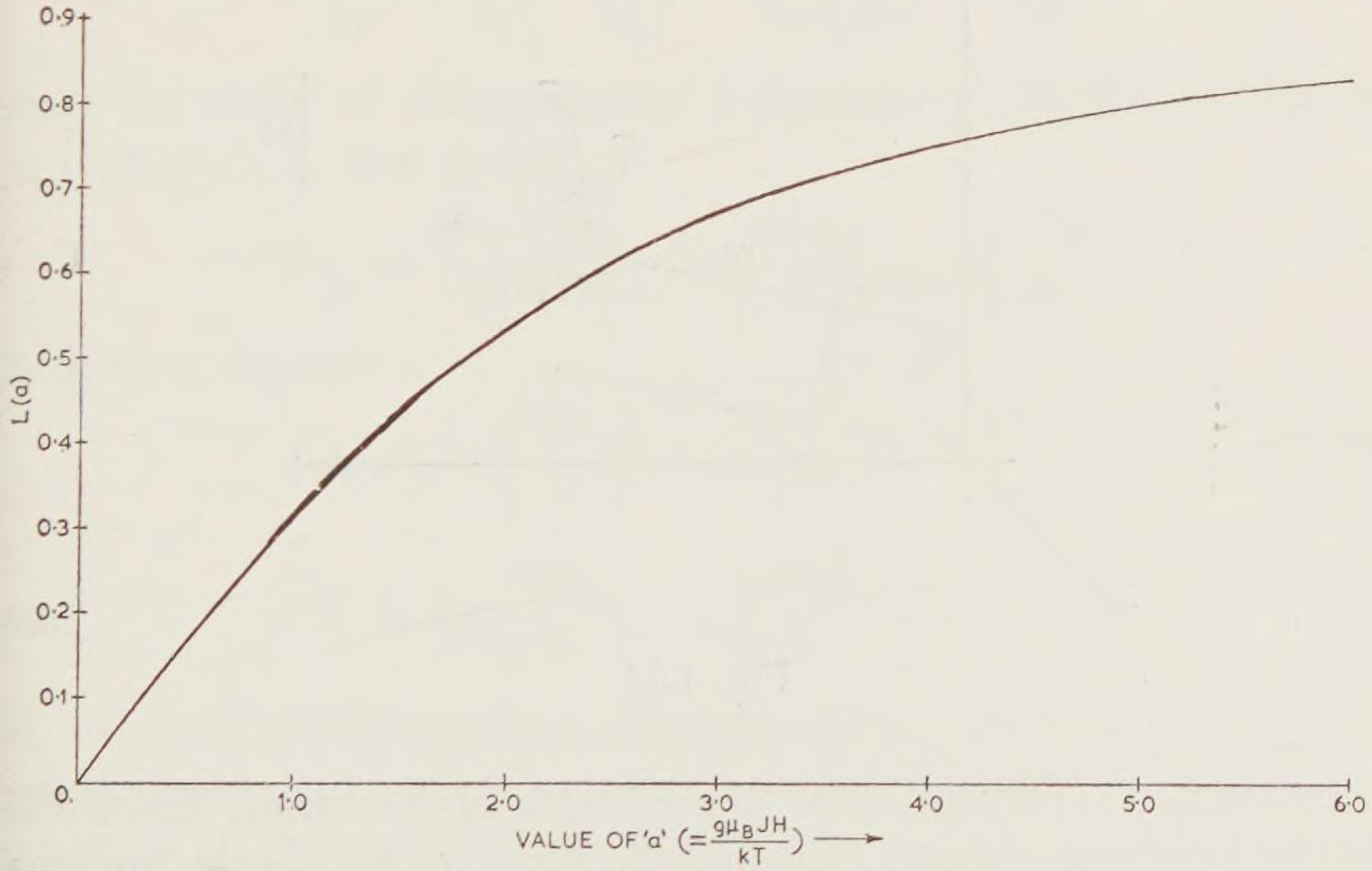


FIG. 1.12

Measurement of Susceptibility

If a magnetic dipole is placed in a non-uniform field it experiences a translational force which is determined by the field and by the susceptibility of the material. This is the basis of most methods of measurement of susceptibility.

If the field is considered to act in some arbitrary direction at a point x, y, z, then we may write its value at a point x_1, y_1, z_1 in terms of its components at x, y, z in the following way

$$H_{x_1} = H_x + \left(\frac{\partial H_x}{\partial x}\right)(x_1 - x) + \left(\frac{\partial H_x}{\partial y}\right)(y_1 - y) + \frac{\partial H_x}{\partial z}(z_1 - z) \quad (1.72)$$

and similarly for H_{y_1} and H_{z_1}.

If we suppose the distance from x, y, z to x_1, y_1, z_1 to be the length of the dipole, a, then $x_1 - x = a_x$, $y_1 - y = a_y$ and $z_1 - z = a_z$. Now we have seen that, in a uniform field, the dipole experiences a torque, illustrated by Fig. 1.12. The torque arises from the force components F_1, perpendicular to the length of the dipole. There will also be force components F_2, along the length of the dipole, but in a uniform field these are equal and opposite.

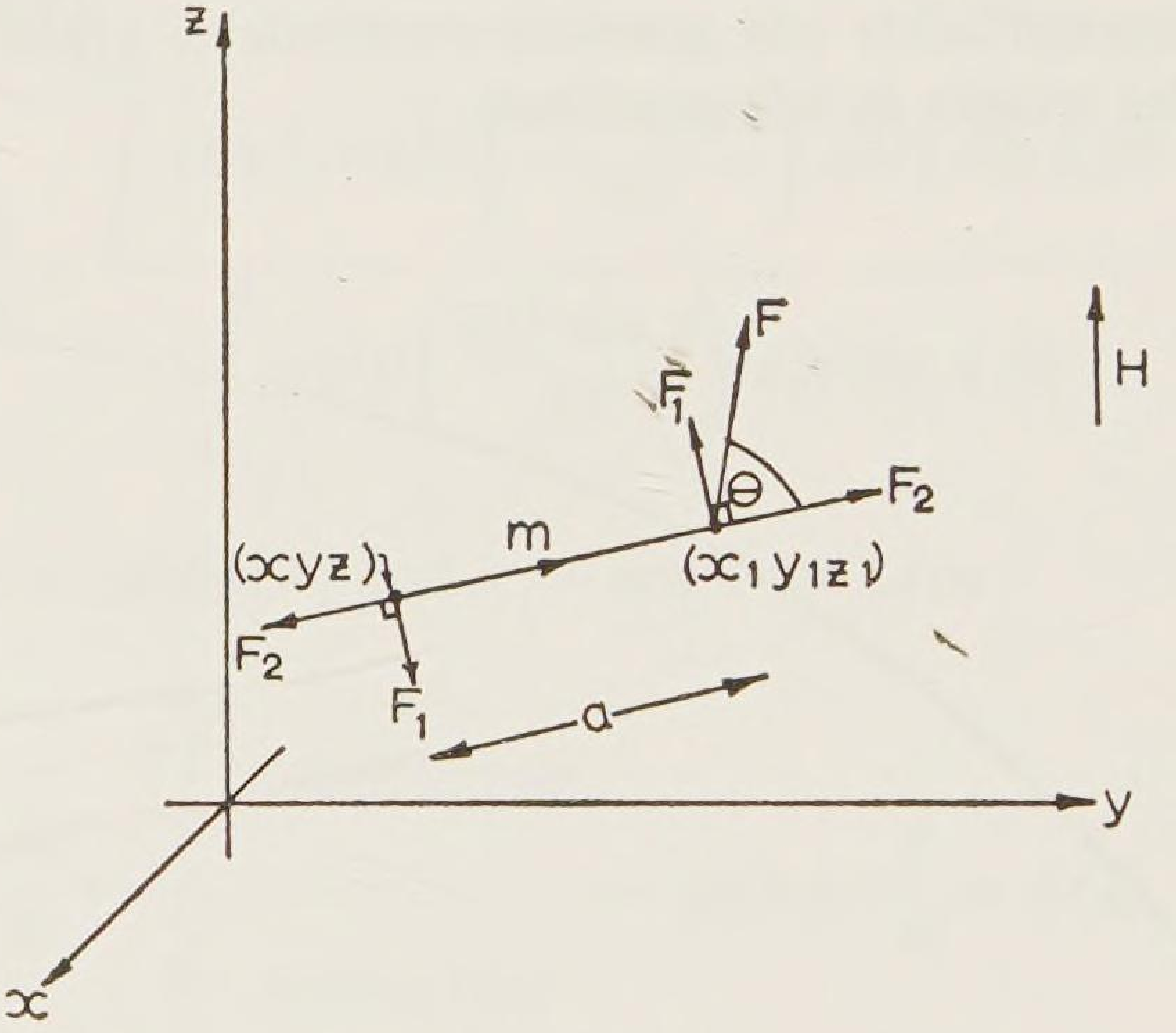


FIG. 1.13

The magnitude of the torque, from equation (1.11), is $L = aF_1 = mH \sin \theta$ and, for the geometry shown in Fig. 1.13, $F_1 = F \sin \theta$ so that $F = (m/a)H$. The component F_2 will clearly be given by $F_2 = (m/a)H \cos \theta$ and, if the field is non-uniform, the value of

H at one end differs from that at the other so that there is a net translational force given by $F_2(x_1y_1z_1)-F_2(xyz)$. Dealing only with the x component and using equation (1.72), the net translational force in the x direction will be given by

$$F_{T_x} = \frac{m}{a}(H_{x_1}-H_x) = \frac{m}{a}a_x\left(\frac{\partial H_x}{\partial x}\right)+\frac{m}{a}a_y\left(\frac{\partial H_y}{\partial x}\right)+\frac{m}{a}a_z\left(\frac{\partial H_z}{\partial x}\right).$$

But (m/a) is the moment per unit length of the dipole, and the vector $\mathbf{m}$ is directed along the length of the dipole so that $(m/a)a_x$ is the component of m in the x direction and $(m/a)a_y$ and $(m/a)a_z$ are the components of m in the y and z directions. Thus

$$F_{T_x} = m_x\left(\frac{\partial H_x}{\partial x}\right)+m_y\left(\frac{\partial H_x}{\partial y}\right)+m_z\left(\frac{\partial H_x}{\partial z}\right).$$

Now from equation (1.23) we have $H = -\text{grad}\,\phi$, so that

$$\frac{\partial H_x}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial x}\right) = \frac{-\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial H_y}{\partial x},$$

since the order of differentiation is immaterial. Similarly $(\partial H_x/\partial z) = (\partial H_z)/\partial x)$, so that finally

$$\left.\begin{aligned}
F_{T_x} &= m_x\left(\frac{\partial H_x}{\partial x}\right)+m_y\left(\frac{\partial H_y}{\partial x}\right)+m_z\left(\frac{\partial H_z}{\partial x}\right)\\
&\text{and, by symmetry}\\
F_{T_y} &= m_x\left(\frac{\partial H_x}{\partial y}\right)+m_y\left(\frac{\partial H_y}{\partial y}\right)+m_z\left(\frac{\partial H_z}{\partial y}\right)\\
\text{and}\quad F_{T_z} &= m_x\left(\frac{\partial H_x}{\partial z}\right)+m_y\left(\frac{\partial H_y}{\partial z}\right)+m_z\left(\frac{\partial H_z}{\partial z}\right)
\end{aligned}\right\}\qquad(1.73)$$

If, in place of a permanent magnetic dipole, we have a sample of magnetizable material of susceptibility χ and volume v, then, from equation (1.71), the moment will be

$$\mathbf{m} = \chi\mu_0 v\mathbf{H}$$

so that

$$\begin{aligned}
m_x &= \chi\mu_0 vH\\
m_y &= \chi\mu_0 vH_y\\
m_z &= \chi\mu_0 vH_z.
\end{aligned}$$

Equation (1.73) then becomes

$$F_{T_x} = \chi\mu_0 v\left(H_x\frac{\partial H_x}{\partial x}+H_y\frac{\partial H_y}{\partial x}+H_z\frac{\partial H_z}{\partial x}\right)$$

$$= \frac{\chi\mu_0 v}{2}\left[\frac{\partial}{\partial x}(H_x)^2+\frac{\partial}{\partial x}(H_y)^2+\frac{\partial}{\partial z}(H_z)^2\right]$$

$$= \frac{\chi\mu_0 v}{2}\frac{\partial}{\partial x}(H^2).$$

So that, by measurement of the force and field gradient in the same direction the susceptibility of the material may be found. However, allowance must be made for the medium in which the material is immersed, e.g. air. If the sample has a susceptibility χ_1 and the air χ_2 then

$$F_{T_x} = (\chi_1-\chi_2)\frac{\mu_0 v}{2}\frac{\partial H^2}{\partial x}. \qquad (1.74)$$

It should be noted that, since only the square of field is involved, reversal of the field does not reverse the force.

Measurement of susceptibility is only called for with non-ferromagnetic materials since, in ferromagnetics χ is so large that $\chi \doteqdot \mu_r$ and it is usually permeability which is the property more readily measured. Thus all the methods based on the foregoing calculations are concerned with the measurement of small forces. With a large electromagnet a value of $(\partial H^2/\partial x)$ of $\sim 10^{11}$ amp^2/metre^3 can be obtained with a field of $\sim 10^6$ amp/metre ($\sim$10,000 oersted). Some difficulty is encountered in achieving a constant field gradient over a usefully large volume but, with conical pole tips, constancy over a volume of 1 c.c. or more can be obtained. Taking a sample of 1 c.c. having a susceptibility of 10^{-3} the force, from equation (1.74), will be approximately $\frac{1}{2}.10^{-3}.4\pi+10^{-7}.10^{-6}.10^{11} = 2\pi\times 10^{-6}$ newtons (i.e. $\sim$0·6 dynes) which is equivalent to a mass of $\sim$0·7 milligrams. Thus the apparatus should be able to measure forces of better than, say, 100 microgrammes for small samples.

Many methods have been designed and a thorough account of most of them has been given by Bates [4]. One method which has been used to measure susceptibilities of the order of 10^{-6} combines

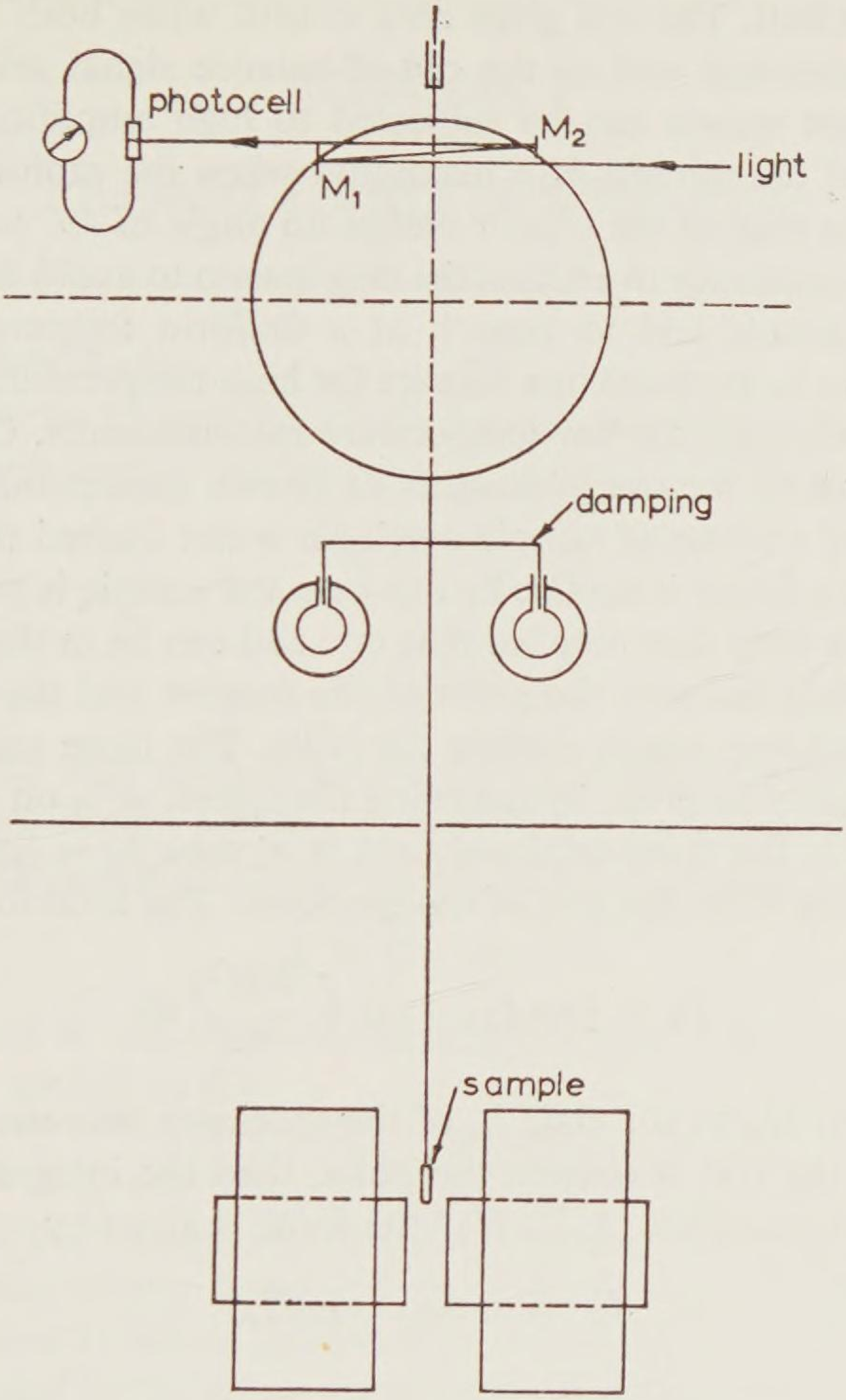


FIG. 1.14

the Sucksmith ring balance with a modern split photocell technique. Referring to Fig. 1.14 a phosphor-bronze ring, of some 10 cm diameter, is made from $\frac{1}{2}'' \times 0\ 01''$ strip. On this are cemented two mirrors M_1 and M_2. The ring is suspended from above and the specimen is hung from it between the poles of the magnet. As the specimen experiences a force the ring distorts and a light beam reflected from M_1 and M_2 is deflected. The deflection is directly proportional to the force and, for high sensitivity measurements, is allowed to fall onto a split photocell. The cell gives zero output when both halves are equally illuminated and so the out-of-balance signal arising when the light spot moves can be subjected to high amplification. The deflection of the mirrors is a maximum when the radius from the centre of the ring to the mirror makes an angle of 49° to the horizontal. It is necessary to enclose the ring system to avoid disturbance from air currents and to keep it at a uniform temperature. The specimen can be enclosed in a furnace for high-temperature measurements or a cryostat for low-temperature measurements. Calibration is carried out by the use of samples of known susceptibility.

Where the volume of sample available is not limited the method originated by Gouy is useful. In this case the sample is packed, if a powder, in a long thin tube, so that one end can be in the region of maximum field between the poles of the magnet and the other in a virtually field-free region outside the poles. The force on the specimen will clearly be given by summing the forces, dF_x, on each small volume δv. If the cross-sectional area is α, then $\delta v = \alpha dx$ where x is the distance from one end of the specimen. The total force will be

$$F_x = \tfrac{1}{2}\alpha\mu_0(\chi_1 - \chi_2) \int \frac{\partial(H^2)}{\partial x}\, dx$$

If the field is H_A at the end, A, of the specimen between the poles, and H_B at the end B outside the poles, then the integral is simple $(H_A{}^2 - H_B{}^2)$ and since $H_A{}^2 \gg H_B{}^2$ the force is given by

$$F_x \doteqdot \alpha\mu_0(\chi_1 - \chi_2)H_A{}^2. \tag{1.75}$$

Forces of several milligrammes can easily be obtained in fields of 10^5 A/m (1000 oersted) upwards and are readily measured. Because of the long specimen, however, it is difficult to make reliable measurements over a wide range of temperatures. Several methods of application of this method are described by Bates [4].

CHAPTER 2

Atomic Theory of Magnetism

Gyromagnetic Ratio

In the previous chapter the direct relation between atomic magnetic moment and angular momentum was discussed. This is given by

$$\mathbf{m} = \gamma \mathbf{G} \tag{2.1}$$

where $$\gamma = -\frac{\mu_0 e}{2m_0}$$

m_0 being the 'rest' mass of the electron. Furthermore, when a field is applied to the atom a torque is exerted on the dipole moment, and, since torque is rate of change of angular momentum, we may write

$$\mathbf{L} = \dot{\mathbf{G}} = \mathbf{m} \times \mathbf{H} \tag{2.2}$$

Now **m** and **G** are proportional and collinear from equation (2.1) (since γ is scalar) so that

$$\dot{\mathbf{G}} = \gamma \mathbf{G} \times \mathbf{H} \tag{2.3}$$

Since the product of two vectors is another vector perpendicular to both, $\dot{\mathbf{G}}$ must always be normal to **G** and **H**. Thus, referring to Fig. 2.1, if **G** and **H** are in the plane of the paper at any instant, **G** is perpendicular to the plane, i.e. if **G** is drawn from a fixed point, its tip must move in a circle, so that **G** precesses about **H**. Expanding (2.3) we have

$$G = \gamma[\mathbf{a}_x(G_yH_z - G_zH_y) + \mathbf{a}_y(G_zH_x - G_xH_z) + \mathbf{a}_z(G_xH_y - G_yH_x)]$$

where $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$ are unit vectors. If the field, H, is applied in the z-direction we have

$$\left.\begin{aligned} \dot{G}_x &= \gamma H G_y \\ \dot{G}_y &= -\gamma H G_x \\ \dot{G}_z &= 0 \end{aligned}\right\} \tag{2.4}$$

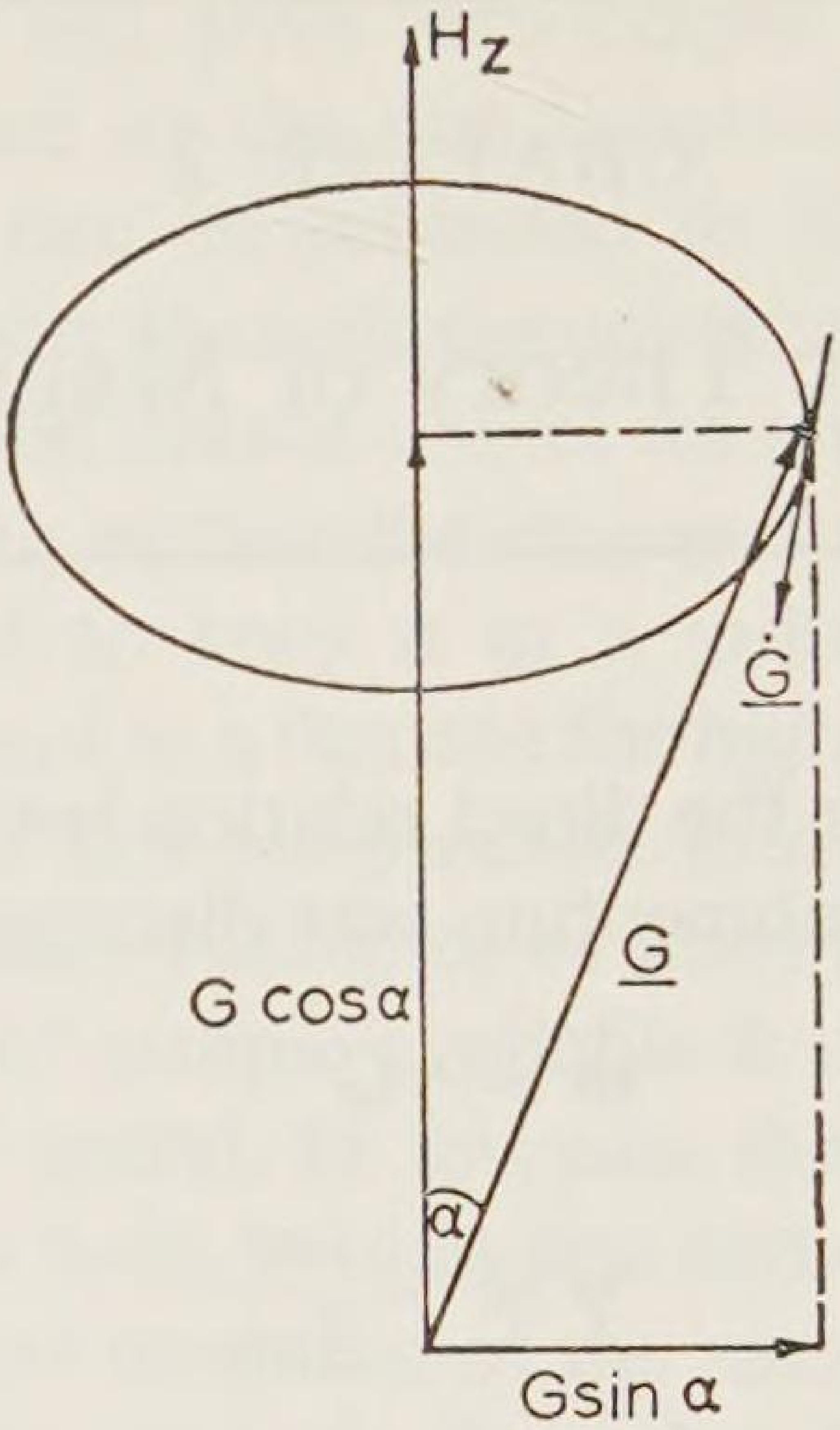


FIG. 2.1

Since (dG_z/dt) is zero, G_z itself must evidently be a constant. Referring to Fig. 2.1, $G_z = G\cos\alpha$, and so the angle α must be constant. Thus the effect of the field is to cause **G** (and hence **m**) to precess about the field direction at a constant angle.

Differentiating eqn. (2.4) we have

$$\ddot{G}_x = \gamma H G_y = \gamma H(-\gamma H G_x)$$

i.e.
$$\ddot{G}_x = -\gamma^2 H^2 G_x. \tag{2.5}$$

Equation (2.5) is the well-known second-order differential equation for which we may write the solution

$$G_x = A\cos(-\gamma H t + \phi)$$

where A is the peak amplitude in the x-y plane and ϕ a constant phase angle; similarly $G_y = A\sin(-\gamma H t + \phi)$.

The peak amplitude of G in the x-y plane is constant and of value $G\sin\alpha$ so that the full solution of equation (2.4) is

$$\left.\begin{aligned} G_x &= G\sin\alpha\cos(\omega_p t + \phi) \\ G_y &= G\sin\alpha\sin(\omega_p t + \phi) \\ G_z &= G\cos\alpha \end{aligned}\right\} \tag{2.6}$$

where $\omega_p = -\gamma H$ is the angular precession frequency depending only on the value of field and the gyromagnetic ratio. The negative sign indicates the direction of the precession, which is that of a right-handed screw along the direction of **H**. Were γH positive the opposite would apply. The value of α is simply the angle between the direction of the dipole moment and the axis along which the field is applied at the moment of switching it on.

The above theory is, of course, over-simplified as may be seen by means of a simple experiment. If a gyroscope is set spinning and hung from a suitable stand on the end of a long thread, it can be made to precess by setting its axis at an angle to the vertical and releasing it. The gyroscope will precess about the vertical, the direction of the gravitational force, which is the analogue of the magnetic field in the case of the atomic gyroscope. However, the precessional motion will be observed to be a spiral with the precessional angle (α in Fig. 2.1) gradually decreasing until the gyroscope hangs vertically. This is due to the frictional damping forces on the gyroscope due to air; if a pointer attached to the lower end of its axis is allowed to dip into a bowl of water, increasing the damping, it spirals into the vertical position more quickly. Equation (2.3) contains no damping term and is therefore incomplete. Various forms of damping term for the atom have been used but the most illustrative is that proposed by Landau and Lifshitz. Equation (2.3) is re-written as

$$\dot{\mathbf{G}} = \gamma \mathbf{G}\times\mathbf{H}+\lambda\mathbf{G}\times(\mathbf{G}\times\mathbf{H}) \qquad (2.7)$$

where λ is a constant. The full solution of this equation is beyond the scope of the present book, but the damping term can be understood in a general way. The vector product $\mathbf{G}\times\mathbf{H}$ is another vector perpendicular to both—call it **A**. Then the product $\lambda\mathbf{G}\times\mathbf{A}$ is yet another vector perpendicular to the plane containing **G** and **A**. Reference to Fig. 2.1 reveals that this vector must be directed inwards towards the H-axis and, since it is proportional to $\dot{\mathbf{G}}$, it means that the precession must follow a spiral path which diminishes α and tends to make **G** align itself with **H**.

The time it takes for the atomic gyroscope to align with the field depends upon the damping factor which is determined by the environment of the atom. When interaction between electron spin and the crystal lattice is the dominant source of damping, the relaxation time is of the order of 10^{-7} sec at $-195°$C and 10^{-2} secs at $4{\cdot}2°$K. If spin-spin interaction predominates the relaxation time is 10^{-9} to 10^{-10} sec, independent of temperature. It should not be forgotten that the randomizing effect of thermal agitation, as shown by the Langevin calculation, is a great deal more important. Without it the atomic moments would spiral in to complete alignment with even the smallest fields in a finite time and magnetic saturation would always occur in any field.

All the foregoing has made the implicit assumption that the atomic angular momentum may have any value. This is known not to be true since angular momentum of the electron in the atom and of the atom itself is quantized and may only take certain discrete values. Equally the Langevin calculation of Chapter 1 assumes that the magnetic moment may take up any orientation and value but, since it is directly related to angular momentum, it too must be quantized. We must therefore proceed to examine the effects of quantum theory on our model of the magnetic behaviour of the atom.

Quantization

The angular momentum of an electron in its orbit is quantized in units of $\hbar(=h/2\pi)$ where h is Planck's constant. The orbital quantum number is l which may have any integral value from 0 to $(n-1)$, where n is the principal quantum number of the atom. The actual magnitude of the orbital angular momentum is given by $[l(l+1)]^{\frac{1}{2}}\hbar$ with $l = 0, 1, 2$ etc. Readers unfamiliar with these ideas should consult an elementary textbook on atomic theory, such as Hume-Rothery [5].

Using this we may repeat the calculation of Chapter 1, in which the magnetic moment of the orbital electron was determined. The magnetic moment is given by the area of the orbit multiplied by the current to which it is equivalent: Area of orbit $= \pi a^2$ where $a =$ orbital radius; current $= \mu_0 \times \text{charge} \times \text{frequency of rotation} = (-e\omega/2\pi)\mu_0$, and angular momentum $= m_0 a^2 \omega = [l(l+1)]^{\frac{1}{2}}\hbar$

therefore $m = \mu_0 \pi a^2 \left(\frac{-e\omega}{2\pi}\right) = \frac{-ea^2}{2} \hbar [l(l+1)]^{1/2} \mu_0$

i.e. $$m = \mu_0 \left(\frac{-e}{2m_0}\right) [l(l+1)]^{1/2} \hbar = -\mu_0 \left(\frac{e\hbar}{2m_0}\right) [l(l+1)]^{1/2} \qquad (2.8)$$

Comparing (2.8) with (1.58), we may write

$$\mathbf{m} = -\mu_0 \left(\frac{e\hbar}{2m_0}\right) \mathbf{l} \qquad (2.9)$$

where $\hbar \mathbf{l} = \mathbf{G}$ and $\mathbf{l}$ is a vector, normal to the plane of the orbit, representing the angular momentum in units of $\hbar$ and having a length proportional to $[l(l+1)]^{\frac{1}{2}}$. Thus m is also quantized.

For the lowest possible value, other than zero, of angular momentum we have $l = 0$ and the magnitude of m is just $(\mu_0 e/2m_0)\hbar$. This is taken as the 'natural' or fundamental unit of magnetic moment and is named the 'Bohr magneton' with the symbol β. For the electron $\beta = (\mu_0 e\hbar/2m_0) = 1{\cdot}165 \times 10^{-29}$ weber-metre $= 0{\cdot}9273 \times 10^{-20}$ e.m.u.

In an applied field the dipole moment precesses about the field direction at a constant angle α (neglecting damping) and the components of angular momentum and of magnetic moment in the direction of the field must also be quantized. The angular momentum component is written as $m_l \hbar$ where m_l is the *magnetic orbital quantum number* and may take all the values from $-l$ to $+l$, including 0. The component of magnetic moment in the field direction will be $-m_l \beta$, the negative sign indicating that angular momentum and magnetic moment are oppositely directed.

The fact that the moment in the direction of the field is quantized means that only certain values of the angle α will be possible such that

$$[l(l+1)]^{1/2} \hbar \cos \alpha = m_l \hbar$$

where l and m_l are integral. This is illustrated in Fig. 2.2 for the case of $l = 3$.

There will also be a rotating component of magnetic moment in the plane perpendicular to the field. It is of magnitude $[l(l+1) - m_l)]^{\frac{1}{2}} \beta$ and rotates with the angular frequency of precession i.e. $-\gamma H$.

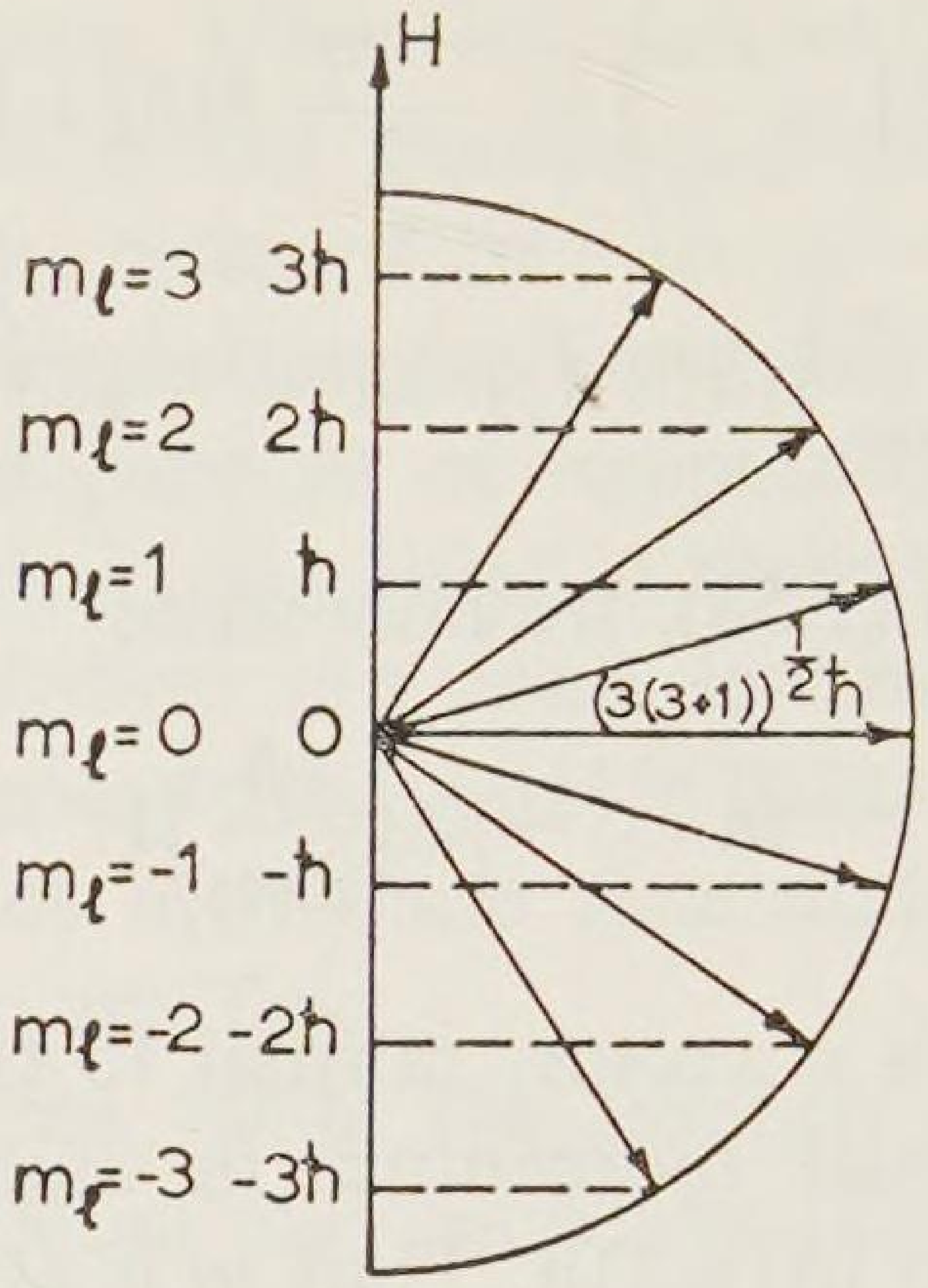


FIG. 2.2

In Chapter 1 it was pointed out that electron spin also contributes to the angular momentum of the electron in the atom and to its magnetic moment. Spin angular momentum is quantized in units of $\hbar$ in the same way. Thus the spin angular momentum is $[s(s+1)]^{1/2}\hbar$ where s is the spin quantum number and is half integral. For a single electron $s = \pm\frac{1}{2}$. However, the case of spin differs from the orbital one in that the ratio of magnetic moment to angular momentum is not the same. To allow for this we write

$$m = -g\mu_0 \frac{e\hbar}{2m_0}(s(s+1))^{1/2} = -g\beta(s(s+1))^{1/2} \qquad (2.10)$$

where g has a value of 2 for a single electron spin, and is known as the Landé splitting factor. Incidentally g can be included in the orbital expression, equation (2.8), because, as will be shown later, it has a value of one for orbital motion.

As in the case of orbital motion we can write, for spin,

$$\mathbf{m} = -g\beta\mathbf{s} \qquad (2.11)$$

where $\mathbf{s}$ is a vector of length $(s(s+1))^{1/2}$ directed along the axis of spin and antiparallel to the resulting magnetic moment $\mathbf{m}$. There will be a quantized component of spin momentum along the field direction of value $m_s\hbar$ where $m_s = \pm\frac{1}{2}$ and the component of magnetic moment will be $m_s g\beta = \mp\frac{1}{2}.2.\beta = \mp\beta$ for a single electron spin.

The Pauli Exclusion Principle states that no two electrons may have the same quantum numbers n, l, m_l and m_s. Since l can have all integral values from 0 to $(n - 1)$, m_l all integral values from $-l$ to $+l$, including 0, and $m_s = \pm\frac{1}{2}$, it follows that $2n^2$ electrons can have a given value of n. Such a collection of $2n^2$ is referred to as a closed shell and such a shell must have as many positive m_l values as negative, so that the total orbital magnetic moment, from equation (2.9), will be zero. Similarly if two electrons contribute $m_s = +\frac{1}{2}$ and $m_s = -\frac{1}{2}$ respectively the total magnetic moment due to spin will also be zero. Thus we conclude that closed shells will have zero magnetic moment and free atoms with unfilled shells will be the only ones capable of producing a net moment.

In considering how the magnetic moments add together in a many-electron atom with unfilled shells, it is important to know how the electrons interact with each other. We now consider the interaction energies arising between electrons.

Coulomb Interaction and Exchange Energy

Electrons, since they carry like charges, repel each other in accordance with Coulomb's law of electrostatic force. In wave mechanics the electrons are represented by a wave-function ψ which has the property that the square of its modulus, $|\psi|^2$, represents the distribution of charge round the nucleus. Coulomb interaction arises when wave-functions overlap and is proportional to the product $|\psi_1|^2|\psi_2|^2$. It is usual to write this as $(\psi_1{}^*\psi_1).(\psi_2^*\psi_2)$, where the asterisk denotes complex conjugate, since the product of a complex number and its conjugate is equal to the square of its modulus, i.e.

$$(a+jb)(a-jb) = a^2+b^2 = [\sqrt{(a^2+b^2)}]^2$$

If we have an atom with an electronic wave function $\psi_1(\mathbf{R})$ where $\mathbf{R}$ is a position vector and a second atom with wave function $\psi_2(\mathbf{R})$ then, when they are brought close together, we may write the total electronic wave-function as

$$\Psi = \psi_1(\mathbf{R}_1)\psi_2(\mathbf{R}_2) - \psi_2(\mathbf{R}_1)\psi_1(\mathbf{R}_2)$$

where the first atom is in the position denoted by suffix 1 and the second atom is in the position denoted by suffix 2. This leads to an expression for the total energy over a volume τ which includes the following terms:

$$\int \psi_1^*(1)\psi_2^*(2)\psi_1(1)\psi_2(2)\frac{e^2}{r_{12}}\,d\tau \tag{a}$$

$$-\int \psi_1^*(2)\psi_2^*(1)\psi_1(1)\psi_2(2)\frac{e^2}{r_{12}}\,d\tau \tag{b}$$

$$-\int \psi_1^*(1)\psi_2^*(2)\psi_1(2)\psi_2(1)\frac{e^2}{r_{12}}\,d\tau \tag{c}$$

$$+\int \psi_1^*(2)\psi_2^*(1)\psi_1(2)\psi_2(1)\frac{e^2}{r_{12}}\,d\tau \tag{d}$$

where (1) and (2) represent positions described by R_1 and R_2 and r_{12} is the distance between positions (1) and (2). Now the terms (a) and (d) contain terms of the type $\psi_1^*(1)\psi_1(1)$ and $\psi_1^*(2)\psi_1(2)$ and therefore represent Coulomb interaction energy. (b) and (c), however, are not of this form and represent some other type of energy which is given the name *Exchange Energy*. It is generally written in the form

$$w = -2J_K\mathbf{s}_i\,.\,\mathbf{s}_j \tag{2.12}$$

where $\mathbf{s}_i$ and $\mathbf{s}_j$ are the spin vectors of electrons i and j and J_K is the 'exchange integral'. This is given by

$$J_K = \iint \psi_a^*(1)\psi_b^*(2)\psi_a(2)\psi_b(1)\,V_{ab}\,d\tau_{12} \tag{2.13}$$

where atoms are a and b and electrons are 1 and 2, as illustrated in Fig. 2.3. The potential V_{ab} is given by

$$V_{ab} = e^2\left(\frac{1}{r_{ab}}-\frac{1}{r_{a2}}-\frac{1}{r_{b1}}+\frac{1}{r_{12}}\right) \tag{2.14}$$

where the symbols are illustrated in Fig. 2.3.

The sign of J_K depends on that of V_{ab} which depends, amongst other things, on the inter-atomic distance r_{ab}. J_K is usually negative which means that (2.12) will be positive unless $\mathbf{s}_i$ and $\mathbf{s}_j$ are oppositely directed. Thus, when the exchange integral is negative, minimum exchange energy is obtained with anti-parallel spins. Slater put J_K

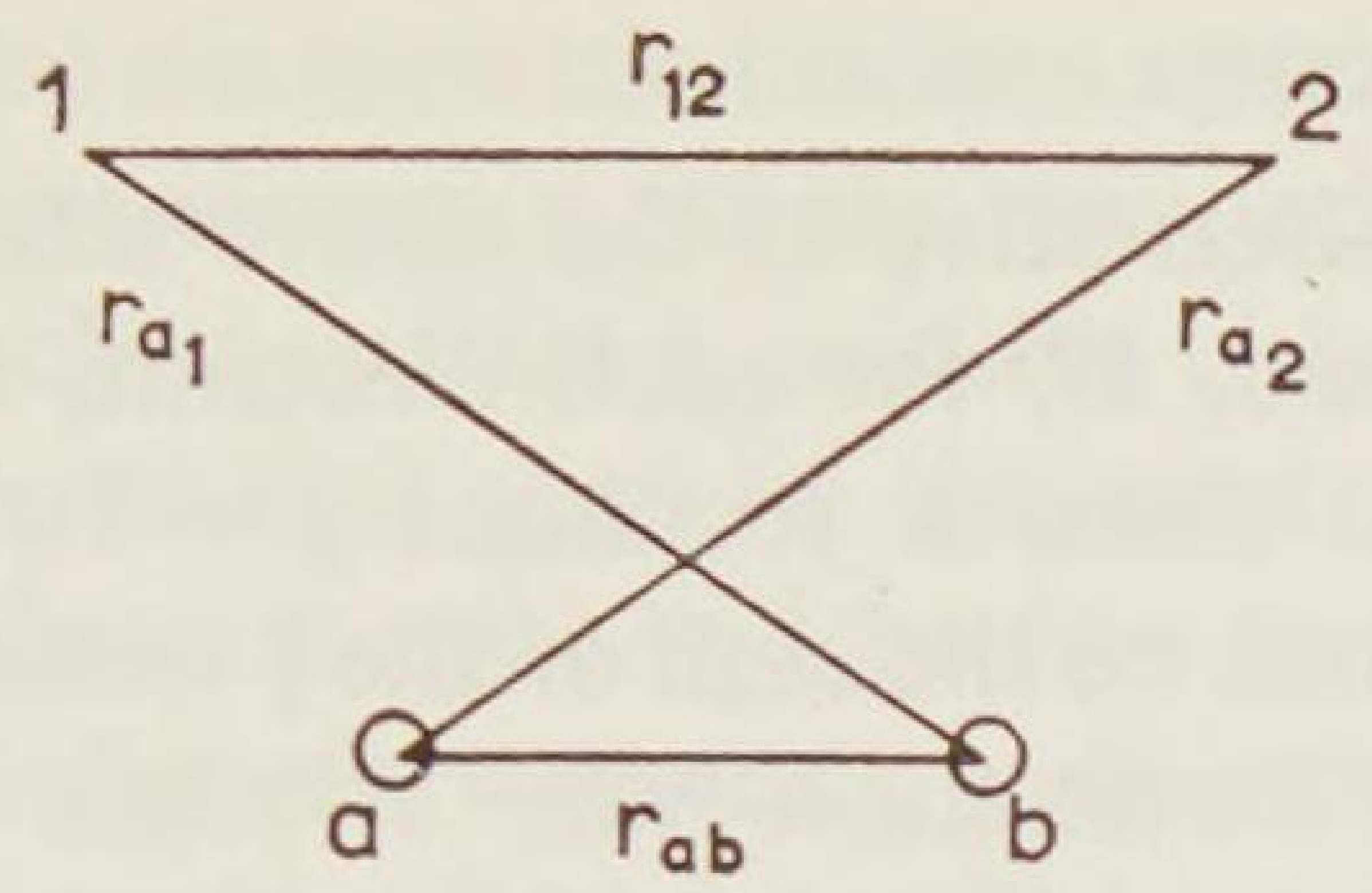


FIG. 2.3

in terms of interatomic distance D and radius, d, of the unfilled shell and showed that, if it is plotted as a function of D/d, J_K is negative until $D/d>3$ when it becomes positive. Minimum energy will then be obtained with the spin vectors parallel to each other indicating a net spin magnetic moment for the atoms. It was a success of the Slater theory that D/d turns out to be greater than 3 for the ferromagnetic materials Fe, Co, Ni and Gd so that the spontaneous magnetization in these materials can be ascribed to a positive exchange integral. This will be considered further in a later section.

If the exchange energy is evaluated between electrons on the *same* atom it is found that J_K is positive. This can be seen in a crude sort of way from (2.14) since $r_{ab} \to 0$ and so $1/r_{ab} \to +\infty$; V_{ab} must be positive and the spins should be parallel. Putting this in general terms for the many-electron atom it means that the s-vectors must add together to give at total resultant spin vector **S** for the atom, which has the maximum possible value consistent with the Exclusion Principle for an atom in the lowest possible energy state (ground state). The same forces will act to couple together the orbital momentum vectors **l** to give a resultant **L** for the whole atom which also has the maximum possible value consistent with the Exclusion Principle.

This is known as Russell-Saunders coupling of the spin and orbital momenta. The total spin momentum of the atom is a vector **S** whose length is $(S(S+1))^{1/2}\hbar$ and the total angular momentum is a vector **L** having the length $(L(L+1)^{1/2}\hbar$. The total resultant angular momentum of the atom is represented by a vector **J** where

$$\mathbf{J} = \mathbf{L}+\mathbf{S} \tag{2.15}$$

whose length is $(J(J+1))^{1/2}\hbar$.

Spin-Orbit Interaction

There is a possible alternative to the above scheme of coupling spin and angular momenta. If, for each electron, the spin and orbital momenta couple to form a resultant $\mathbf{j} = \mathbf{l}+\mathbf{s}$, the total atomic momentum would then be the sum of the **j** vectors. This would mean that the spin-orbit interaction was stronger than the exchange interaction and such cases are only found in the free atoms of some of the heavy elements. For the present purposes, therefore, only Russell-Saunders coupling will be considered so that the spin-orbit interaction will be between the total **L** and total **S** vectors giving the total momentum vector **J**.

There is a further implication of spin-orbit coupling for which it is first necessary to refer back to the question of quantum numbers. It will be recalled that no two electrons may have all four quantum numbers the same. Thus, for example, if we consider the case where $n = 2$, $l = 1$, m_l may have all values from $-l$ to $+l$, including 0 and m_s may be $\pm\frac{1}{2}$. There are the following possibilities:

m_l	m_s
$+1$	$+\frac{1}{2}$
0	$+\frac{1}{2}$
-1	$+\frac{1}{2}$
$+1$	$-\frac{1}{2}$
0	$-\frac{1}{2}$
-1	$-\frac{1}{2}$

so that six electrons constitute the closed shell for which $n = 2, l = 1$. Now we have seen that exchange forces favour parallel orientation of spin so that, if there were just three electrons, they would have values $m_s = +\frac{1}{2}$ and $m_l = -1$, 0 and $+1$ with the consequence that the net orbital angular momentum in a weak magnetic field given by $\sum_{-1}^{+1} m_l \hbar$, will be zero. This argument can be generalized, from which emerges the fact that a half-full shell will have zero total orbital angular momentum (i.e. $L = 0$) in the direction of an applied field.

Now consider a single electron orbiting round the nucleus. From the point of view of the electron the nucleus appears to be circulating around it and it experiences a weak magnetic field due to the nucleus. Since the nucleus carries a positive charge this field will be opposite in direction to that due to the electron orbit. Now the moment due to the electron spin will be free to align itself with the nuclear field and will therefore assume a direction opposite to that of the orbit and the vectors $\mathbf{l}$ and $\mathbf{s}$ will be antiparallel. Thus, in combining $\mathbf{L}$ and $\mathbf{S}$ for a shell which is less than half-full the resultant is given by $\mathbf{J} = \mathbf{L}-\mathbf{S}$. For a shell which is half-full $\mathbf{L} = 0$, $\mathbf{J} = \mathbf{S}$, and spin-orbit coupling is inoperative. When the shell is more than half-full the electrons in the incomplete half-shell again contribute to spin-orbit coupling and, since $\mathbf{S}$ now must be reversed in sign in accordance with the Exclusion Principle, the resultant will be given by $\mathbf{J} = \mathbf{L}+\mathbf{S}$.

All the foregoing is summarized by Hund's rules as follows:

(i) The spins s_i combine to form a resultant S which has the maximum possible value consistent with the Exclusion Principle.
(ii) The orbital vectors l_i combine to form a resultant L which has the maximum value consistent with the Exclusion Principle and with rule (i).
(iii) The resultants L and S combine to form J, the value of which is $L-S$ for a less-than-half-full shell and $L+S$ for a more-than-half-full shell.

We may illustrate this with reference to a $3d$ shell for which $n = 3$ and $l = 2$. The shell can contain just ten electrons of which there will be five with $m_s = +\frac{1}{2}$, $m_l = +2, +1, 0, -1, -2$ and similarly five with $m_s = -\frac{1}{2}$. If the shell is incomplete and contains three electrons they will all have $m_s = +\frac{1}{2}$ so that $S = 3/2$ and $L = 2+1+0 = 3$. For the half-full shell containing five electrons $S = 5/2$ and $L = 2+1+0-1-2 = 0$. For six electrons $S = 5/2-\frac{1}{2} = 2$, $L = 2$ and for seven $S = 5/2-2/2 = 3/2$ and $L = 2+1 = 3$. The resultants will be, for three electrons, $J = L-S = 3-3/2 = 3/2$, for five electrons $J = S = 5/2$, for six electrons $J = L+S = 2+2 = 4$ and for seven electrons $J = 3+3/2 = 9/2$.

In general, the energy associated with coupling between **L** and **S** is given by

$$W = \lambda_{so}\mathbf{L}.\mathbf{S} \tag{2.16}$$

where λ_{so} is the spin-orbit coupling parameter, being positive for shells less than half-full, zero for a half-full shell and negative for shells more than half-full.

Magnetic Moment of a Free Atom

In the previous section the way in which the spin and orbital angular momenta combine to give the total atomic angular momentum has been discussed. Since magnetic moment is antiparallel and proportional to angular momentum the net magnetic moment can therefore be found in a similar way. However account must be taken of the fact that the factor of proportionality between magnetic moment and angular moment is $-g\beta$ where $g = 1$ for orbital motion and $g = 2$ for spin.

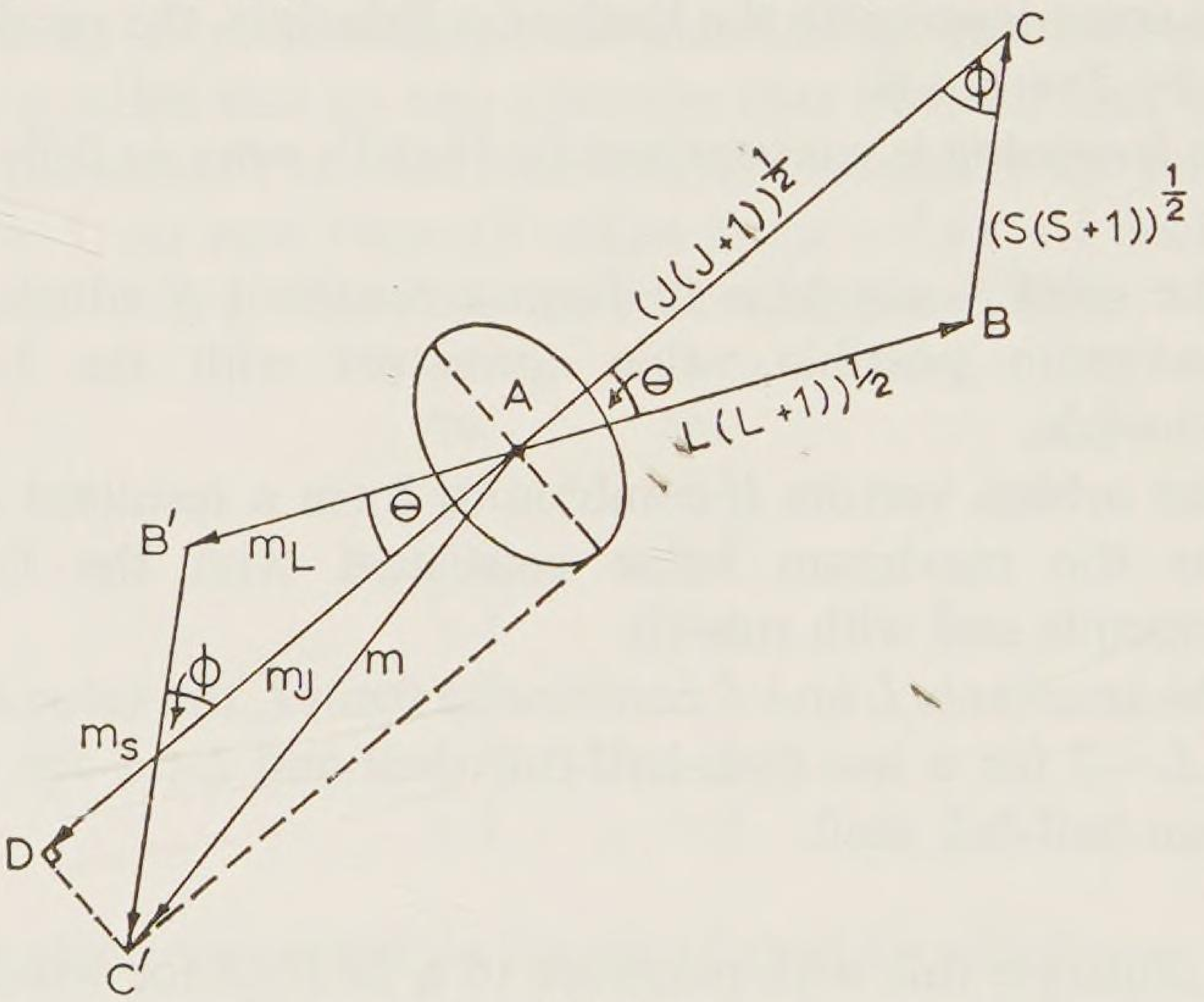


FIG. 2.4

A vector diagram may be drawn to represent the situation, as in Fig. 2.4, in which the lengths of the vectors J, L and S are, respectively $(J(J+1))^{1/2}$, $(L(L+1))^{1/2}$ and $(S(S+1))^{1/2}$ and the associated moments m_J, m_L and m_S are antiparallel to J, L and S and are proportional to their lengths. The vectors $AB(=\mathbf{L})$ and $BC(=\mathbf{S})$ add to give $AC(=\mathbf{J})$. AB' is antiparallel to AB and represents the orbital magnetic moment, m_L, whilst $B'C'$ is antiparallel to BC and represents m_S but has twice the proportionality factor that exists between L and m_L. The result of adding m_L and m_S is the total magnetic moment m which is not antiparallel to J. Now m (and L) must precess, in the field due to m_S, about the direction of m_S and, similarly, m_S (and S) must precess in the field due to m_L about the direction of m_L. Since there is no external force on the atom the total angular momentum vector J must be fixed in space, thus the total magnetic moment vector m, equal to $-g\beta(J(J+1))^{1/2}$, must precess about the direction of J. It will have a component m_J along the direction of J and this will equal the sum of the components of m_L and m_S projected on the J-direction. Referring to the diagram

$$\begin{aligned} m_J &= m_L \cos\theta + m_S \cos\phi \\ &= -\beta(L(L+1)^{1/2})\cos\theta - 2\beta(S(S+1))^{1/2}\cos\phi \end{aligned}$$

Using the trigonometrical cosine formula in triangle ABC

$$-\cos\theta = \frac{(S(S+1)) - L(L+1) - J(J+1)}{2(L(L+1)J(J+1))^{1/2}}$$

$$\therefore \qquad m_L \cos\theta = \frac{\beta(S(S+1) - L(L+1) - J(J+1))}{2(L(L+1))^{1/2}}$$

Similarly

$$-\cos\phi = \frac{(L(L+1) - S(S+1) - J(J+1))}{2(S(S+1)J(J+1))^{1/2}}$$

$$\therefore \qquad -m_S \cos\phi = \frac{2\beta(L(L+1) - S(S+1) - J(J+1))}{2(J(J+1))^{1/2}}$$

Thus

$$m_J = \frac{\beta L(L+1) - S(S+1) - 3J(J+1)}{2(J(J+1))^{1/2}}$$

If we define the moment associated with J as

$$m_J = -g\beta(J(J+1)^{1/2}$$

where g is the Landé splitting factor used previously, then

$$g = \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)}. \tag{2.17}$$

It is seen from equation (2.17) that, when $L = 0$,

$$g = \frac{3}{2} + \frac{S(S+1)}{2S(S+1)} = 2$$

whereas if $S = 0$

$$g = \frac{3}{2} - \frac{L(L+1)}{2L(L+1)} = 1.$$

Thus, if the magnetic moment of the free atom is measured, the value of g may be determined and will indicate the proportions of the total moment due to orbital and to spin motion of the electrons.

In an applied field the total momentum J will precess about the field direction and there will be a component of it $M_J \hbar$ in this direction, where M_J is the total magnetic quantum number and can take all values from $+J$ to $-J$ including zero giving a total of $(2J+1)$ possible values. The total magnetic moment, in the field direction, will be given by

$$M = -g\beta M_J \tag{2.18}$$

The Zeeman Effect

If an atom is excited so that it emits light, the spectral lines of the light correspond to transitions between the allowed energy levels of the atom. When a magnetic field is applied, the lines, if viewed perpendicular to the direction of the field, are seen to split into three components, one of higher and one of lower frequency, and the original line. This is known as the Zeeman effect and may be understood by reference to the preceding paragraph. If the atom has a magnetic moment of $-g\beta M_J$ in the field direction it will have a potential energy in the field of $g\beta M_J . H$. This represents the introduction of $(2J+1)$ energy levels with the $M_J = 0$ condition corresponding to the original one. M_J varies from $-J$ to $+J$, and the extra levels will be separated from the original level by an amount proportional to the field. The wavelength separation of the Zeeman levels is very small being of the order of 0·1Å in a field of 1 weber/m^2 (10,000 gauss).

Thus, for an incomplete shell with $J = \frac{1}{2}$ there will be two additional energy levels, one above the original level, corresponding to $M_J = +\frac{1}{2}$, and one below, with $M_J = -\frac{1}{2}$. For $J = 3/2$ there would be four additional levels with $M_J = 3/2, \frac{1}{2}, -\frac{1}{2}$ and $-3/2$. It will be seen that measurement of the Zeeman splitting provides a method of determining the magnetic moment of an atom.

The experimental determination is, however, not so straightforward as the above suggests. In most substances the Zeeman splitting is more complex than that described above. If the transition giving rise to a particular spectral line is between energy levels E_1 and E_2, then the frequency of the emission, ν, is given by

$$h\nu = E_1 - E_2.$$

When a field H is applied each energy level splits into a number of levels given by $E_1 + g_1\beta M_{J1}$ and $E_2 + g_2\beta M_{J2}$ where g_1 and M_{J1} are the Landé splitting factor and magnetic quantum number appropriate to the atom in the energy state E_1 and g_2, M_{J2} to energy state E_2. Thus the emitted frequencies are given by the equation

$$h\nu = E_1 - E_2 + (g_1 M_{J1} - g_2 M_{J2})\beta H$$

subject to the selection rule that M_J may only change by 0 or ± 1. If $g_1 = g_2 = 1$, as for simple orbital magnetic moment, and $J = \frac{1}{2}$, there would be just three lines, but since this is not usually the case and $g_1 \neq g_2$ in most materials there are frequently more than the expected three lines. This was referred to as the 'anomalous' Zeeman effect, but the anomaly was really lack of the quantum theory to explain it at the time when it was first observed.

The Barnett and Einstein-de Haas Effects

It has been seen that the gyromagnetic ratio for the electron differs according to whether orbital or spin momentum is the source of magnetic moment. For the orbital case $g = 1$ and for spin $g = 2$; thus an experiment designed to measure the g-factor will indicate which of spin or orbital motion of the electrons is the principal source of magnetic moment in the material concerned.

One such experiment makes use of the fact that, if a rod of magnetic material is spun rapidly about its longitudinal axis, it acquires a magnetization. This is known as the Barnett [6] effect, after its discoverer, and it can be understood by reference to the ideas on precession already introduced.

When a field is applied, the atomic magnetic moment precesses about its direction with an angular velocity ω_p, producing a net magnetization in the direction of the field. From simple magnetostatic theory, if the atom has a magnetic moment M which makes an angle α with the field direction, there will be a torque on the atomic dipole, tending to align it with the field, given by

$$L = -MH \sin \alpha \tag{2.19}$$

From the precession theory of equation (2.4) this torque is given by $\dot{G}_y$ (or $\dot{G}_x$) and may be written as

$$L = -\gamma HG \sin \alpha = \omega_p G \sin \alpha \tag{2.20}$$

where ω_p is the precession frequency.

Equating (2.19) and (2.20) we have

$$-MH = \omega_p G$$

or

$$\frac{G}{M} = -\frac{H}{\omega_p}.$$

Experimentally the rod is rotated at an angular frequency ω, which represents an externally impressed precession, and the resulting magnetization is measured. The rod is then demagnetized and a field, H, applied until the same degree of magnetization is obtained. This then gives H/ω and hence G/M, the gyromagnetic ratio, which is $1/\gamma$.

For the electron

$$\frac{1}{\gamma} = -\frac{2}{g}\left(\frac{m}{\mu_0 e}\right) \qquad (2.21)$$

so that the measurement leads to evaluation of g. For the ferromagnetic materials and for most paramagnetics, g is found to have a value near to 2. Thus it may be concluded that the principal source of magnetism is electronic spin, with only a small contribution from orbital moment.

The Einstein-de Haas effect is the inverse of the Barnett effect. A rod of magnetizable material is freely suspended in a coil providing a field. When the field is suddenly reversed, reversing the magnetization, there is a mechanical reaction, due to reversal of the direction of the axes of the atomic gyroscopes, which causes the rod to rotate on its suspension. Measurement of this rotation permits evaluation of the gyromagnetic ratio. For full details the reader is referred to a paper by Scott [7].

The Curie Law

In Chapter 1 the average moment of an assembly of atoms was calculated on the basis of classical statistics, leading to the Langevin function. Although this calculation accounts in a general way for the observed temperature-dependence of susceptibility, experimentally it agrees only with measurements on some gases. A closer fit with experiment is obtained by including the quantization of magnetic moment discussed in the preceding sections of this Chapter.

Following the procedure used in Chapter 1 we use the Boltzmann expression for the probability of an atom having an energy W, i.e. $\exp(-W/kT)$. Now the energy of an atomic dipole in a field H is given by $g\beta M_J H = W$ so that the fraction of atoms with an energy W is given by

$$dn = \frac{\exp(-g\beta M_J H/kT)}{\sum_{M_J} \exp(-g\beta M_J H/kT)}$$

(assuming all atoms to be in the same energy state and this to be the ground state).

The component of magnetic moment parallel to H is $-g\beta M_J$ and therefore the total moment of a system of n atoms is

$$M = n\bar{m} = \frac{m\sum\limits_{-J}^{+J}(-g\beta M_J)\exp(-g\beta M_J H/kT)}{\sum\limits_{-J}^{+J}\exp(-g\beta M_J H/kT)}$$

Let $g\beta H/kT = x$, then

$$M = ng\beta\,\frac{\sum\limits_{-J}^{+J} M_J\exp(M_J x)}{\sum\limits_{-J}^{+J}\exp(M_J x)} \tag{2.22}$$

$$= ng\beta\,\frac{d}{dx}\left(\log\sum_{-J}^{+J}\exp(M_J x)\right)$$

$$= mg\beta\,\frac{d}{dx}\left\{\log\left(\exp(Jx)\,\frac{(1-\exp[-(2J+1)x])}{1-\exp(-x)}\right)\right\}$$

$$= ng\beta\,\frac{d}{dx}\left[\log\left(\frac{e^{(J+\frac{1}{2})x}-e^{-(J+\frac{1}{2})x}}{e^{x/2}-e^{-x/2}}\right)\right]$$

$$= ng\beta\,\frac{d}{dx}\left[\log\left(\frac{\sinh(J+\frac{1}{2})x}{\sinh\frac{1}{2}x}\right)\right]$$

i.e.

$$M = ngJ\beta\left[\frac{2J+1}{2J}\coth\left(\frac{2J+1}{2J}\right)a-\frac{1}{2J}\coth\left(\frac{a}{2J}\right)\right] \tag{2.23}$$

where

$$a = \frac{gJ\beta H}{kT}$$

The expression in the square brackets in equation (2.23) is called the Brillouin function, denoted by B_z, and describes the variation of the induced magnetization with temperature. When J becomes large B_z tends to the Langevin function [coth $a-(1/a)$] as may be expected since the large number of levels, corresponding to large J, approaches an energy continuum.

As discussed in Chapter 1 '*a*' will normally be much less than 1 so that the exponentials in equation (2.22) can be replaced by the first two terms of their expansion.

Thus

$$M = n\bar{m} = -ng\beta\frac{\sum\limits_{-J}^{+J} M_J\left(1-\frac{g\beta M_J H}{kT}\right)}{\sum\limits_{-J}^{+J}\left(1-\frac{g\beta M_J H}{kT}\right)}$$

$$= \frac{ng^2\beta^2 H}{(2J+1)kT}\sum_{-J}^{+J} M_J^2$$

$$= \frac{ng^2\beta^2 H}{(2J+1)kT}\frac{1}{3}J(J+1)(2J+1)$$

i.e.
$$M = \frac{ng^2\beta^2 HJ(J+1)}{3kT} \tag{2.24}$$

The susceptibility, χ, is given by

$$\chi = \frac{M}{\mu_0 H} = \frac{ng^2\beta^2 J(J+1)}{3kT\mu_0} = \frac{C}{T} \tag{2.25}$$

which is in agreement with the Curie Law given in Chapter 1, but with the Curie constant given by $[ng^2\beta^2 J(J+1)]/3k\mu_0$ instead of $nm^2/3k\mu_0$. However, for large fields at low temperatures the condition $a \ll 1$ no longer applies and the full B_z expression must be used.

A plot of B_z as a function of '*a*' is given for $g = 2, J = \frac{1}{2}, 3/2, 5/2$, in Fig. 2.5. The function $L(a)$ is shown in the same graph for comparison. Experiments by Henry [8] on selected paramagnetic salts have shown good agreement between the Brillouin expression and the measured temperature and field dependence of magnetization, in fields up to 10^7 amps/metre (~50 k.gauss) and temperatures down to 1·3°K.

Susceptibility of Solids

The paramagnetic salts used by Henry were selected on the basis that they contained ions with incomplete shells which would behave,

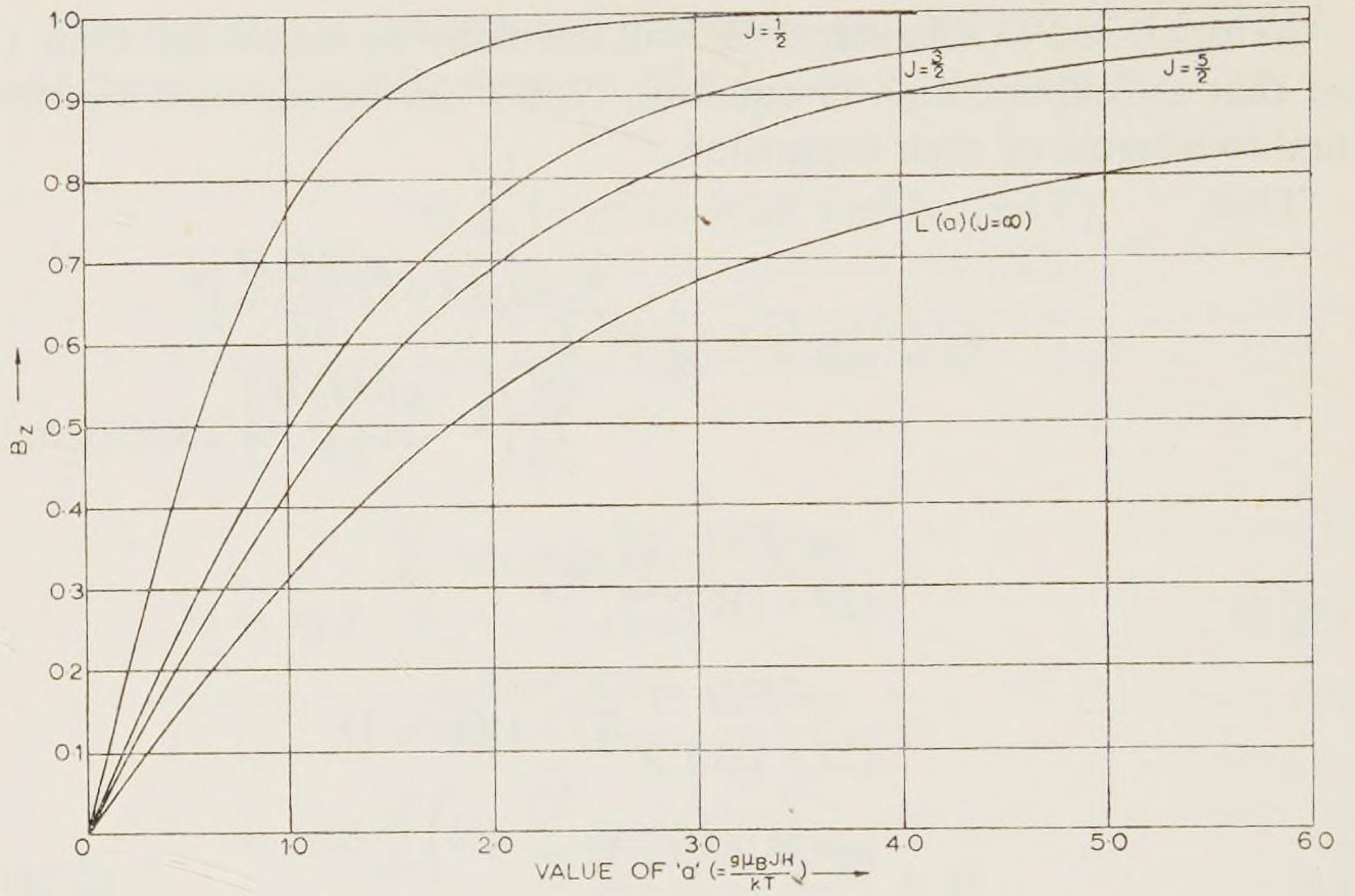


FIG. 2.5

in the salt, more or less as free atoms. In most solids the atoms are subject to coupling effects and fields which considerably affect their magnetic behaviour. It was mentioned, in connexion with equation (2.14), that the exchange forces between the electrons on different atoms are generally such as to favour antiparallel alignment of spins, resulting in zero spin moment whenever possible. Equally the orbits tend to be arranged, within a molecule, to give a net zero orbital moment so that the molecules tend to have a zero total magnetic moment. There will always, of course, be a diamagnetic susceptibility as discussed previously. The only common gases showing paramagnetic behaviour are oxygen, in which the electron spins are unpaired and the moment corresponds to $s = 1$, $g = 2$, and nitrous oxide, NO, in which there is an odd number of electrons, resulting in one unpaired spin.

In crystalline solids the type of bonding between the atoms is a determining factor in whether or not there is a net magnetic moment. In the ionic crystal, such as NaCl, electrons are transferred between atoms to form closed shells which have, of course, no net moment. In normal covalent bonding, such as is met in diamond, germanium, silicon, etc., the bonds contain electron pairs which have antiparallel spins and again no net moment. Metallic bonding is something of a special case. In most metals the ionic cores show only diamagnetism, but the 'free' conduction electrons show a temperature-independent paramagnetism, which is discussed in the next section.

The crystal structure of a solid corresponds to a certain 'crystal electric field' whose symmetry determines or is determined by the orientation of the outer electron orbitals, concerned with bonding the atoms together. The orbitals can be regarded as locked in position or 'quenched', so that they cannot be reoriented by an applied magnetic field. This means that if a net magnetization is produced by a field, it is due to reorientation of the electron spins. The point can be illustrated by reference to some experimental results.

The Transition Elements

Elements in which incomplete shells occur are called transition elements, and correspond, as one progresses through a group of them, to the gradual filling up of an electron shell. Whichever is the shell being filled, an ion with a given number of electrons in the unfilled shell always shows much the same magnetic properties, as may be expected from the application of Hund's rules, even though the ions are formed from different atoms. The groups are: the iron group, in which the $3d$ shell ($n = 3$, $l = 2$) is being filled, the palladium group ($4d$ shell, $n = 4$, $l = 2$), the lanthanide group ($4f$ shell, $n = 4$, $l = 3$), the platinum group ($5d$ shell, $n = 5$, $l = 2$) and the actinide group ($5f$ shell, $n = 5$, $l = 3$). Bleaney and Bleaney [9] have summarized the experimental results for the iron group in a table, which is given as Table 2.1 below. In this the spectroscopic notation is used in column 3, in which the superscript gives the number of electrons in the incomplete shell, the letters S, P, D, F correspond to $L = 0, 1, 2, 3$, respectively, and the subscript gives the value of J. The experimental results are given in terms of p^2 where the square of the measured magnetic moment per atom, m^2, is $p^2\beta^2$ and p is called the effective Bohr magneton number per atom. Referring to equation (2.25) we see that, for an assembly of free ions $p^2 = g^2J(J+1)$. The salts chosen for the measurements included in Table 2.1 are all magnetically dilute, which means that the paramagnetic ions are sufficiently far apart ($\geqslant$6Å) that they do not interact significantly and the Curie law of susceptibility is obeyed. In the last column the value of $g^2S(S+1)$ is given, with $g = 2$, corresponding to the spin-only value for p^2. In the second from last column is given $g^2J(J+1)$ with g calculated from equation (2.17). It will be seen that the experimental results agree far better with the last column, suggesting that, in most cases, the orbital contribution to the total moment is largely quenched and the moment is not that which would be obtained with an assembly of free ions. It will be seen that Fe^{2+}, Co^{2+} and Ni^{2+} give poorest agreement with the spin-only value, the experimental value of p^2 always being high, indicating a contribution due to appreciable spin-orbit coupling.

TABLE 2.1

No. of electrons in 3d	Ion	Ground state	S	L	J	p^2 (expt)	$g^2J(J+1)$	$4S(S+1)$
0	$K^+Ca^{2+}Sc^{3+}$	$^1S_0$	0	0	0	0	0	0
1	$Ti^{3+}V^{4+}$	$^2D_{3/2}$	$\frac{1}{2}$	2	3/2	2·9	2·4	3
2	V^{3+}	$^3F_2$	1	3	2	6·8	2·67	8
3	$V^{2+}Cr^{3+}$	$^4F_{3/2}$	3/2	3	3/2	14·8	0·6	15
4	$Cr^{2+}Mn^{3+}$	$^5D_0$	2	2	0	(23·3)	0	24
5	$Mn^{2+}Fe^{3+}$	$^6S_{5/2}$	5/2	0	5/2	34·0	35	35
6	Fe^{2+}	$^5D_4$	2	2	4	28·7	45	24
7	Co^{2+}	$^4F_{9/2}$	3/2	3	9/2	24·0	44	15
8	Ni^{2+}	$^3F_4$	1	3	4	9·7	31·3	8
9	Cu^{2+}	$^2D_{5/2}$	$\frac{1}{2}$	2	5/2	3·35	12·6	3
10	Cu^+Zn^{2+}	$^1S_0$	0	0	0	0	0	0

Pauli Spin Paramagnetism

In a metal which is a good conductor the conduction electrons approximate, in their behaviour, to a gas of free electrons, independent of the parent atoms. We have seen that the spinning electron has a magnetic moment of one Bohr magneton and the application of a field will cause the electrons to acquire a magnetic potential energy $\beta H \cos \theta$, where θ is the angle between the spin axis and the field. For a spin parallel to the field the potential energy will be $-\beta H$ and for one antiparallel to the field the energy will be $+\beta H$. The antiparallel spins will tend to turn to parallelism with the field with an accompanying change of energy of $2\beta H$.

The implications of this may be seen by considering the density of states curve of Fig. 2.6(a), in which the number of energy states available to an electron in the free electron gas is plotted as a function of electron energy. The graph has been divided into two halves, one showing the density of states available for electrons with positive spin (upwards, say) and one for electrons with negative (downwards) spin. The Fermi energy, $E_F(0)$, is shown and represents

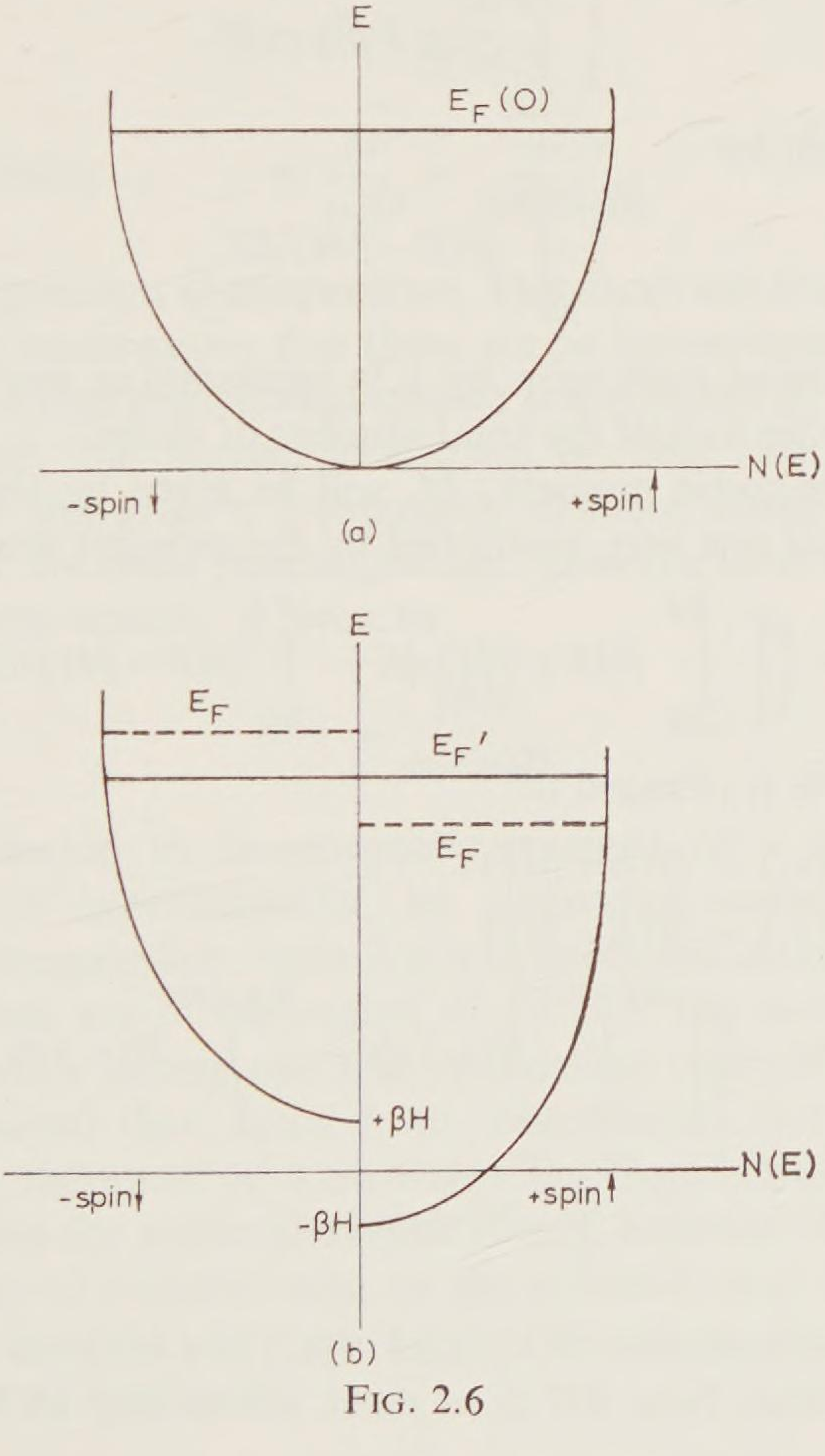


FIG. 2.6

the energy up to which all the available states are filled by electrons at the absolute zero of temperature.

When a field, H, is applied the energy distribution is altered, as in Fig. 2.5(b), and there will be a spill-over of electrons from $-$spin to $+$spin until a new Fermi level, E_F', is established at some value differing by a small amount from the original energy $E_F(0)$. Taking the temperature to be 0°K, the total number of electrons is given by adding up all the states of each spin between energies 0 and E_F'. In this case the zero for $N(E)$ is at $-\beta H$ for $+$spin and at $+\beta H$ for $-$spin. The total number of electrons with $+$spin is therefore given by

$$\frac{1}{2}\int_{-\beta H}^{E_F'} N(E+\beta H)\,dE,$$

and for $-$spin by

$$\frac{1}{2}\int_{\beta H}^{E_F'} N(E-\beta H)\,dE.$$

Multiplication of each term by $\frac{1}{2}$ is necessary as each half of the diagram applies to half the total number of states.

The net magnetic moment, M, will be given by the number of excess spins of one sign, multiplied by the moment per spin.

$$\text{i.e.} \qquad M = \frac{\beta}{2}\left[\int_{-\beta H}^{E_F'} N(E+\beta H)\,dE - \int_{+\beta H}^{E_F'} N(E-\beta H)\,dE\right]$$

If the variable is changed to

$$N(\epsilon_+) = N(E+\beta H)$$

and

$$N(\epsilon_-) = N(E-\beta H)$$

and

$$M = \frac{\beta}{2}\left[\int_0^{(E_F'+\beta H)} N(\epsilon_+)\,d\epsilon_+ - \int_0^{E_F'-\beta H} N(\epsilon_-)\,d\epsilon_-\right] \qquad (2.26)$$

$$= \frac{\beta}{2}\int_{E_F'-\beta H}^{E_F'+\beta H} N(\epsilon)\,d\epsilon, \qquad (2.26)$$

since the two functions $N(\epsilon_+)$ and $N(\epsilon_-)$ are both of precisely the same form, $N(\epsilon)$. Now βH is, at most, about 10^{-3} eV for a field of

10^7 amps/metre (100,000 oersted) and the Fermi level is several eV for most solids; we may take $E_F' \doteqdot E_F(0)$ and the integration is simply the area of a strip of width $2\beta H$ centred around $E_F(0)$. This area is $2\beta HN(E_F(0))$ so that the net magnetic moment in the direction of the field is given by

$$M = \beta^2 HN(E_F(0)) \tag{2.27}$$

Now $N(E_F(0))$, the number of states per unit volume in energy range dE, is given by $3N/[2E_F(0)]$ at absolute zero, where N is the number of electrons per unit volume (see, for instance, Kittel [10] p. 258).

Thus
$$M = \beta^2 H \frac{3N}{2E_F(0)}$$

and susceptibility
$$\chi = \frac{M}{\mu_0 H} = \frac{3\beta^2 N}{2\mu_0 E_F(0)} \tag{2.28}$$

which is independent of temperature. This result was first obtained by Pauli on the assumptions that there are no interactions between the electrons and that their spatial motion is unaffected by the magnetic field. Actually the motion is affected by field and Landau [11] showed that this gives a diamagnetic moment whose value is just one-third of the Pauli paramagnetism. Thus the total susceptibility, neglecting interactions, is given by

$$\chi = \frac{\beta^2 N}{\mu_0 E_F(0)} \tag{2.29}$$

This contribution to paramagnetic moment of a solid may be separated out experimentally, by measuring susceptibility as a function of temperature, since it is a temperature-independent term. Typical values are in the region of 5×10^{-6} per metre^3 in m.k.s. units for metals having one free conduction electron per atom. It should be noted that, apart from contributions due to spin-spin interactions, there will be a diamagnetism from bound electrons in the ionic cores for which allowance should be made. Also $N(E_F(0))$ is a function of effective mass of the electrons and, if this differs appreciably from the rest mass, m_0, as it does in the alkali metals, its effect should be included.

CHAPTER 3

Ferromagnetism

Exchange Energy and the Molecular Field

In the previous chapter attention has been concentrated on the behaviour of free atoms, the magnetic properties arising from exchange interaction between electrons on the same atom. We now return to the case of exchange effects between elecrons on adjacent atoms. Using equation (2.11) for exchange energy and applying it to atoms i and j which have total spin vectors $\mathbf{S}_i$ and $\mathbf{S}_j$, the exchange energy is given by

$$w = -2J_K\mathbf{S}_i . \mathbf{S}_j \tag{3.1}$$

Suppose that atom i is surrounded by z equidistant nearest neighbours of the type of atom j, the total exchange energy will be given by

$$w = -2J_K\mathbf{S}_i . \sum \mathbf{S}_j = -2J_K\mathbf{S}_i . (z\overline{\mathbf{S}}_j) \tag{3.2}$$

where $\overline{\mathbf{S}}_j$ is the mean value of spin vector for all the z neighbours.

By definition, the magnetic moment per unit volume of the lattice comprising the atoms j will be

$$\mathbf{M} = ng\beta\overline{\mathbf{S}}_j \tag{3.3}$$

where n is the number of atoms per unit volume. The moment of atom i will be

$$\mathbf{m} = g\beta\mathbf{S}_i \tag{3.4}$$

assuming the magnetic moment to be due to spin only. Combining equations (3.2), (3.3) and (3.4)

$$w = -2J_K\left(\frac{\mathbf{m}}{g\beta}\right) . \left(\frac{z\mathbf{M}}{ng\beta}\right) = \frac{-2J_K z}{ng^2\beta^2}\,\mathbf{m} . \mathbf{M} \tag{3.5}$$

This is the exchange energy and if we treat it as the potential energy of atom **i** in a field due to the lattice of atoms j we may write

$$w = -\mathbf{m}.\mathbf{H}_i \tag{3.6}$$

where

$$\mathbf{H}_i = \frac{2J_K z}{ng^2\beta^2}.\mathbf{M}. \tag{3.7}$$

In this way the exchange energy is treated as an internal field, acting to align the atomic dipoles **m**, whose magnitude is proportional to the total magnetization **M**. Such a field was postulated, *a priori*, by Weiss, who wrote it as

$$\mathbf{H}_i = \lambda\mathbf{M} \tag{3.8}$$

where λ is called the 'molecular field constant'.

In Chapter 2 we considered a paramagnetic substance as an assembly of non-interacting magnetic ions. The idea of internal field takes account of interaction between the ions and it is a reasonable assumption that the Curie law of susceptibility would still apply, but with the modification that the total field acting on the ions will be the sum of the applied and internal fields. Modifying the Curie law in equation (2.24) we have

$$\mathbf{M} = \frac{\mu_0 C(\mathbf{H}_0 + \lambda\mathbf{M})}{T}$$

where $\mathbf{H}_0$ is the applied field and

$$C = \frac{ng^2\beta^2 S(S+1).}{3k\mu_0}$$

Thus the susceptibility is given by

$$\chi = \frac{\mathbf{M}}{\mu_0\mathbf{H}_0} = \frac{C}{T - C\lambda\mu_0} = \frac{C}{T-\theta} \tag{3.9}$$

where $$\theta = \frac{ng^2\beta^2 S(S+1)}{3k\mu_0}.\frac{2J_K z\mu_0}{ng^2\beta^2} = \frac{2J_K zS(S+1)}{3k} \tag{3.10}$$

and is a constant having the dimensions of temperature.

Equation (3.9) represents the temperature-dependence of the susceptibility of most paramagnetic solids quite well, for temperatures greater than θ. At $T = \theta$ the susceptibility becomes infinite and, since magnetization **M** has a finite maximum it requires $H_0 = 0$. Thus we have the possibility of a finite magnetization in the absence of an applied field, i.e. spontaneous magnetization. θ thus emerges as a transition temperature between paramagnetic and ferromagnetic states. It is referred to as the Curie temperature, but it should be noted that the actual transition temperature may differ from θ, in practice, by as much as 20°C. This will be discussed in a later section. Curie temperatures for the ferromagnetic elements are Fe, 1043°K; Co, 1394°K; Ni, 631°K; Gd, 289°K.

The Weiss Theory

Taking nickel as an example we may calculate the value of the exchange integral J_K from the Curie temperature, using equation (3.10). With $S = \frac{1}{2}$ and $\theta = 631$°K we get $J_K \doteqdot 100\,k = 1{\cdot}38 \times 10^{-21}$ joules where k is Boltzmann's constant. The saturation magnetization of Ni is 0·606 Bohr magnetons per atom. Using this, together with the calculated value of J_K and $g = 2$ in equation (3.7), we get $H_i = 10^9$ A/m ($\sim 10^7$ oersted). This is clearly much larger than any applied field capable of realization. We conclude, therefore, that, at temperatures below the Curie temperature, a ferromagnetic material will be spontaneously magnetized to saturation. This means that the magnetization must be represented by the full Brillouin function given by equation (2.23), i.e.

$$M = B_z M_s \tag{3.11}$$

where M is the magnetization at any temperature and M_s is the saturation magnetization. Now the component of magnetic moment of an atom in the field direction is $M_J g\beta$ which has a maximum value, for $M_J = J$, of $Jg\beta$. For the spin-only case $J = S$ so that the saturation magnetization is given by

$$M_s = ng\beta S \tag{3.12}$$

n being the number of atoms per unit volume. For $J = S$ the argument, a, of the Brillouin function can be written, using (3.12), as

$$a = \frac{g\beta HS}{kT} = \frac{M_s H}{nkT} \tag{3.13}$$

where $H = H_0 + \lambda M$.

i.e.
$$a = \frac{M_s(H_0 + \lambda M)}{nkT}$$

from which

$$\frac{M}{M_s} = a\left(\frac{nkT}{\lambda M_s^2}\right) - \frac{H_0}{\lambda M_s} \tag{3.14}$$

and, from (3.11), we also have

$$\frac{M}{M_s} = B_z \tag{3.15}$$

In Fig. 3.1 equations (3.14) and (3.15) are plotted as M/M_s against a, the solid straight lines representing equation (3.14) for different temperatures with $H_0 = 0$. Clearly the point of intersection of the graphs represent simultaneous solutions of the two equations and therefore values for M/M_s. For the line (a) the only intersection is at $M/M_s = 0$ and it corresponds to a temperature for which there is no spontaneous magnetization, i.e. above the transition temperature T_c. For the line (c) there is a finite solution at the point P and spontaneous magnetization exists. The line (b) is tangential to the Brillouin curve at the origin and will correspond to the transition temperature, T_c, at which the spontaneous magnetization appears and disappears. This may be calculated as follows:

Near the origin, 'a' is small so that B_z may be represented by the approximate expression of equation (2.23)

i.e.
$$B_z = \frac{M_0}{M_s} = a\,\frac{(S+1)}{3S} \tag{3.16}$$

where M_0 is the magnetization in the absence of an applied field ($H = H_i$). Using (3.14), with $H_0 = 0$, together with (3.16) we have

$$\frac{a(S+1)}{3S} = a\left(\frac{nkT_c}{\lambda M_s^2}\right) \quad \text{at } T = T_c$$

Thus
$$T_c = \frac{\lambda M_s^2}{nk}\left(\frac{S+1}{3S}\right) = \frac{\lambda n g^2 \beta^2 S(S+1)}{3k} \tag{3.17}$$

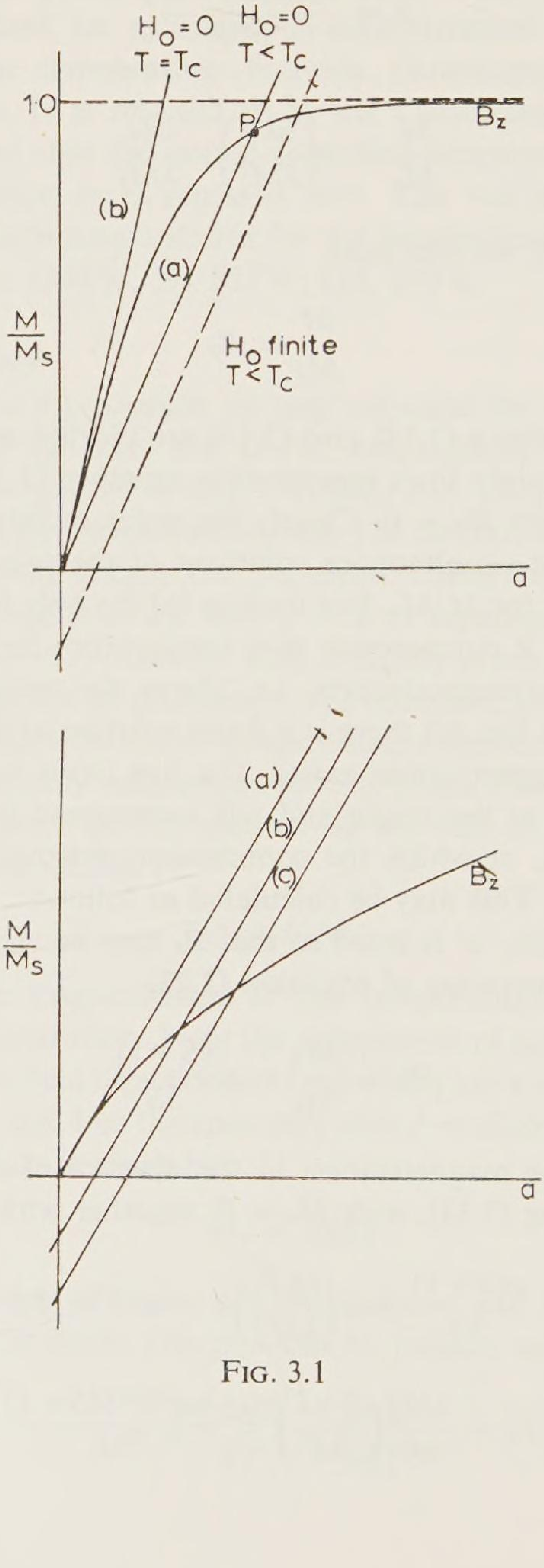

$H_0 = 0$
$T = T_c$
$H_0 = 0$
$T < T_c$
1·0
P
B_z
(b)
(a)
$\frac{M}{M_s}$
H_0 finite
$T < T_c$
a
(a)
(b)
(c)
B_z
$\frac{M}{M_s}$
a

FIG. 3.1

But, from equation (3.7),

$$\lambda = \frac{2J_K z}{ng^2\beta^2}$$

so that

$$T_c = \frac{2J_K zS(S+1)}{3k}$$

and this is just the value of θ given in equation (3.10). Thus θ is identified as the Curie temperature, T_c, at which spontaneous magnetization appears or disappears.

Re-arranging equation (3.17) we have

$$\frac{\lambda M_s^2}{nk} = \frac{3\theta S}{(S+1)}$$

and using this in equation (3.14) it may be re-written as

$$\frac{M_0}{M_s} = a\left(\frac{T}{\theta}\right)\left(\frac{S+1}{3S}\right) \qquad \text{for } H_0 = 0 \qquad (3.18)$$

The implication of (3.18) is that for all materials of the same spin, the magnetization is the same function of T/θ. This is known as the *Law of Corresponding States.* It is illustrated in Fig. 3.2 in which the solid line represents equation (3.18) for the case of $J = \frac{1}{2}$ and experimental points are shown for Ni and Fe.

Referring again to Fig. 3.1 the case of finite applied field, H_0, is shown by the dotted straight line. The field has the effect of shifting the straight line to the right, slightly raising M/M_s and producing a negative intercept on the M/M_s axis corresponding to $H_0/\lambda M_s$. This provides a means for an approximate determination of the molecular field constant λ,

It should be emphasized immediately that the above theory predicts that any piece of ferromagnetic material will automatically be magnetized to saturation, so long as its temperature is below the Curie point. But it is common experience that a piece of iron, for instance, will remain apparently unmagnetized unless a field is applied to magnetize it. Weiss met this difficulty by postulating the existence of 'domains', which are small volumes of the material containing $10^{17}-10^{21}$ atoms, which are magnetized to saturation.

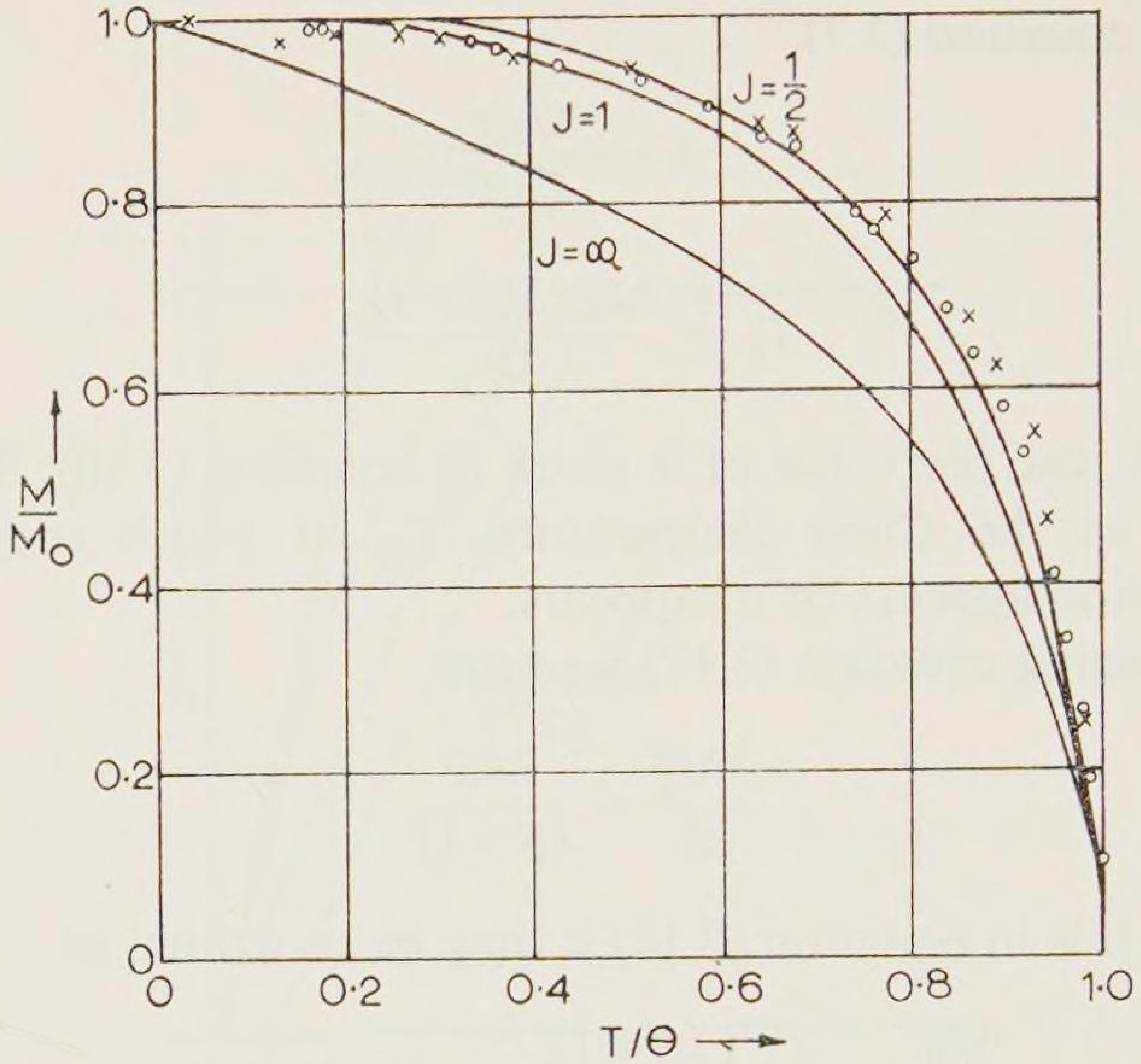


FIG. 3.2

The unmagnetized condition corresponds to random orientation of the domains. This concept has important implications as regards the technical magnetic behaviour of ferromagnetics and is dealt with in a later section.

In the Weiss theory the molecular field is taken to be proportional to the average magnetization throughout the specimen. It is more reasonable to suppose that a given atomic moment will tend to lie parallel with its neighbouring spins rather than to the average magnetization. The formation of clusters of parallel spins in this way could lead to a lowering of the Curie temperature, on the argument that such a cluster could more easily be disturbed by thermal agitation. This would account for differences between the actual observed transition temperature and the temperature θ of the Curie-Weiss law. A theoretical treatment of this case has been given by P. R. Weiss [13].

Collective Electron Theory

So far we have dealt with ferromagnetic materials as an assembly of ions each having a dipole moment, coupled by exchange energy. On the basis of the theory as developed so far, if the saturation magnetization is measured, given by $ngS\beta$, it should give an integral number of Bohr magnetons, β, per atom, if $S = \frac{1}{2}$ and $g = 2$. The actual measured values, for M_s in each of the ferromagnetic elements, are

$$\text{Fe} = 2{\cdot}221\ \beta, \quad \text{Co} = 1{\cdot}716\ \beta, \quad \text{Ni} = 0{\cdot}606\ \beta$$

and $$\text{Gd} = 7{\cdot}10\ \beta,$$

which evidently do not fit the above theory. These values could be accounted for by assuming that, in each case, the material contains atoms in various states of ionization, contributing different magnetic moments. A more useful approach, however, is the collective electron theory, based on the energy-band picture of solids.

We have seen that magnetic moment is associated with unfilled shells in the free atom. When the atoms are aggregated in the solid state a band of permitted energy levels corresponds to each shell. Thus the 4s shell ($n = 4$, $l = 0$) gives a 4s band containing many more permitted levels than the two (corresponding to $n = 4$, $l = 0$, $m_l = 0$, $m_s = \pm\frac{1}{2}$) allowed in the isolated atom. Taking nickel as an example the atomic configuration is such that there is a partially filled 3d-shell with a filled 4s-shell outside it, i.e. the outer electronic configuration is $3d^8 4s^2$. In the solid state the two 4s electrons enter the 4s band and become free conduction electrons, no longer being tied to their parent atoms. This leaves an array of ions each having eight 3d electrons forming partially filled shells. Now the 3d shell is full when occupied by 10 electrons, 5 of positive and 5 of negative spin. Thus, with only eight, we should expect there to be two unpaired electrons, as regards spin, giving a magnetic moment of 2 Bohr magnetons per atom.

However, in the solid, the 3d and 4s energy bands overlap to an appreciable extent so that some of the 4s electrons may occupy energy levels in the 3d band, reducing the effective number of vacancies in this band and hence the average magnetic moment per atom.

This is illustrated diagrammatically in Fig. 3.3. Remembering that exchange energy exists between electrons on the same atom, and that it favours parallel spin alignment, we can represent the density of states in the $3d$ band by two curves, one for positive and one for negative spin. These are displaced along the energy axis, with respect to each other, by an amount equal to the exchange energy. The $4s$ band is not shown displaced since the electrons are free, but even if the two halves of it were displaced it would make little difference since $N(E)$, the number of states, is a slowly varying function of E compared with the $3d$ band. The Fermi levels for Fe

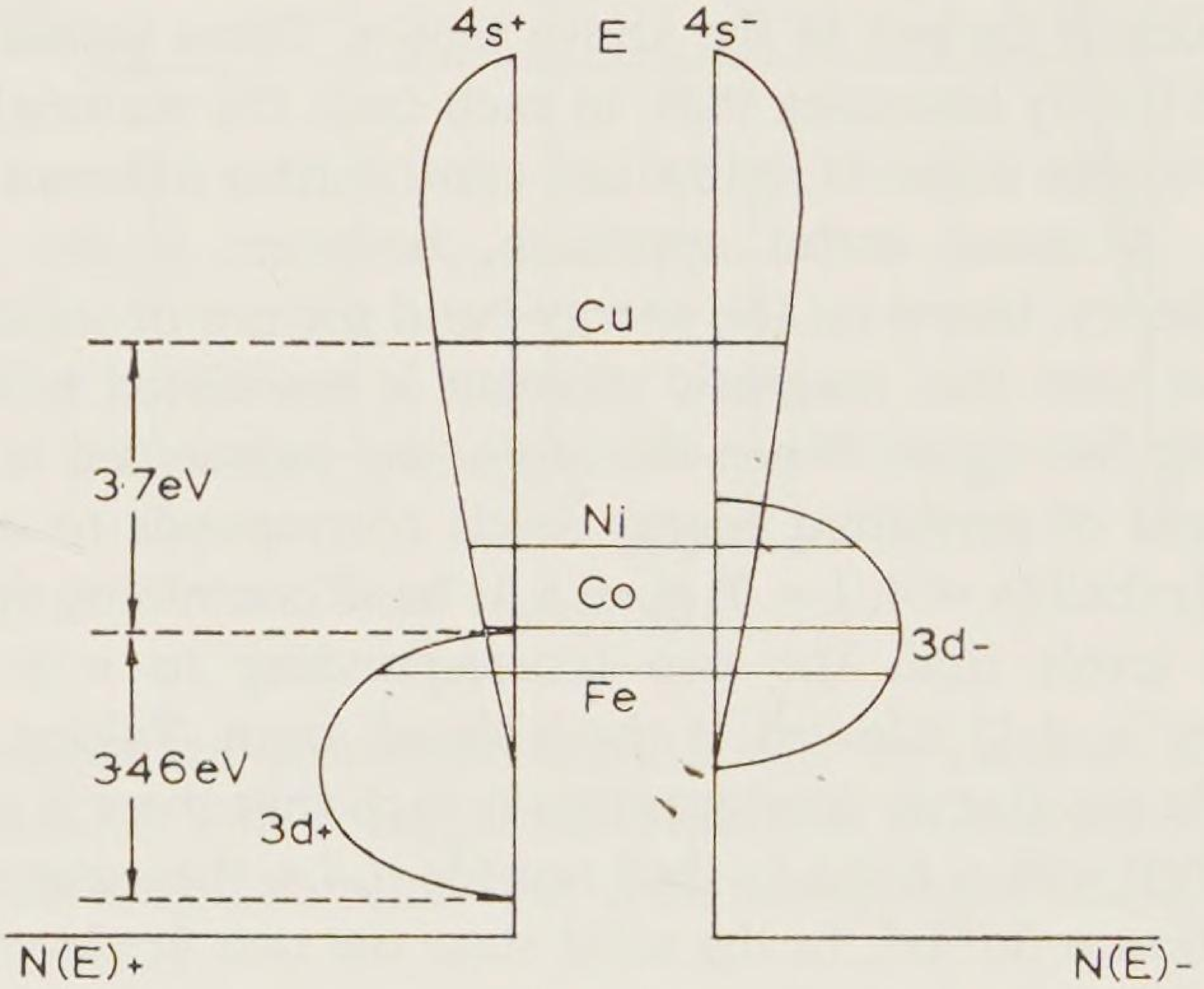


FIG. 3.3

and Ni are shown in the diagram. For Ni its position is such that the $3d+$ band is full and the $3d-$ band has $\sim 0{\cdot}6$ holes per atom, i.e. there are $\sim 0{\cdot}6$ states per atom left unfilled and we should expect a magnetic moment of 0·6 Bohr magnetons per atom. In point of fact experimental measurements show that for nickel, g is somewhat less than 2 due to an appreciable orbital contribution to the moment. If the figure of 0·6 is multipled by $2/g$ we obtain the spin-only contribution to the magnetic moment and this has the value 0·54. Thus the actual figure should be 0·54 holes in the $3d$-band which can be tested experimentally in the following way: Copper has a full $3d$ band, containing 10 electrons, and a single $4s$ electron. If it is alloyed with nickel the $4s$ electron, due to the considerable $4s-3d$ overlap in energy, spends 90 % of its time in the $3d$ band of the nickel thus 'filling up' the holes. When 60 atomic per cent Cu has been added the resulting alloy exhibits zero magnetic moment showing that all the 3*d* holes have been filled by 4*s* electrons from the copper. But 90% of 60% means that there are just 0·54 holes in the *d*-band, in excellent agreement with theory.

The Fermi levels for Fe and Co are also shown in the diagram. For Fe there are 0·2 holes in $3d+$ and 2·4 in $3d-$ giving a net value of 2·2. For Co the corresponding figures are no holes in $3d+$ and 1·7 in $3d-$ giving a net value of 1·7. Gadolinium is not shown as its magnetism arises from holes in the 4*f* rather than the 3*d* band.

Ferromagnetic Domains

We now return to the problem, mentioned earlier, that whilst theory predicts that a ferromagnetic material will be spontaneously magnetized to saturation below the Curie temperature, in fact it can appear to have zero magnetization in the absence of an applied field. Further experimental facts must also be considered. When the material is placed in a field, and the field is varied from a maximum value in one direction to a maximum in the opposite direction, a graph of the flux density in the material (B) against the field (H) shows the well-known hysteresis-loop shape which differs for different materials, as shown in Fig. 3.4. It is convenient to describe these shapes by the parameters B_r, the remanent flux density in zero

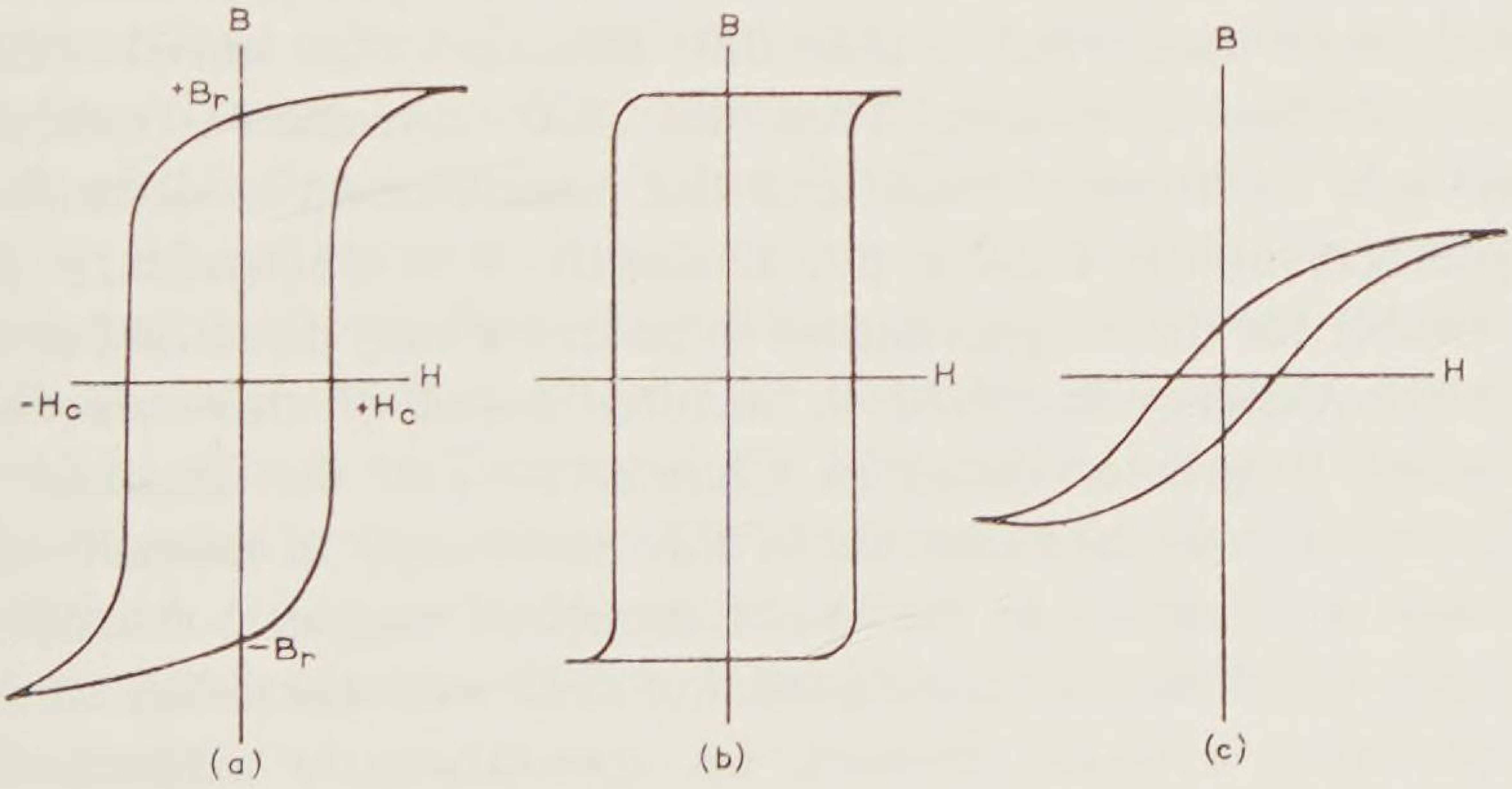


FIG. 3.4

field, and H_c, the coercive force or field required to reduce the remanent flux to zero. Values of B_r can vary from the region of $\sim 0{\cdot}001$ weber/m^2 to ~ 1 weber/m^2 (10–10,000 gauss) and H_c from ~ 200 amp/metre to $\sim 2 \times 10^5$ amp/metre (1–1000 oersted).

The initial magnetization curve, which plots the increase of magnetization from an initially demagnetized state, typically has an S-shape and any theory of magnetization should be able to account for this. It is also found that this curve differs, when the specimen is a single crystal, with differing crystallographic directions.

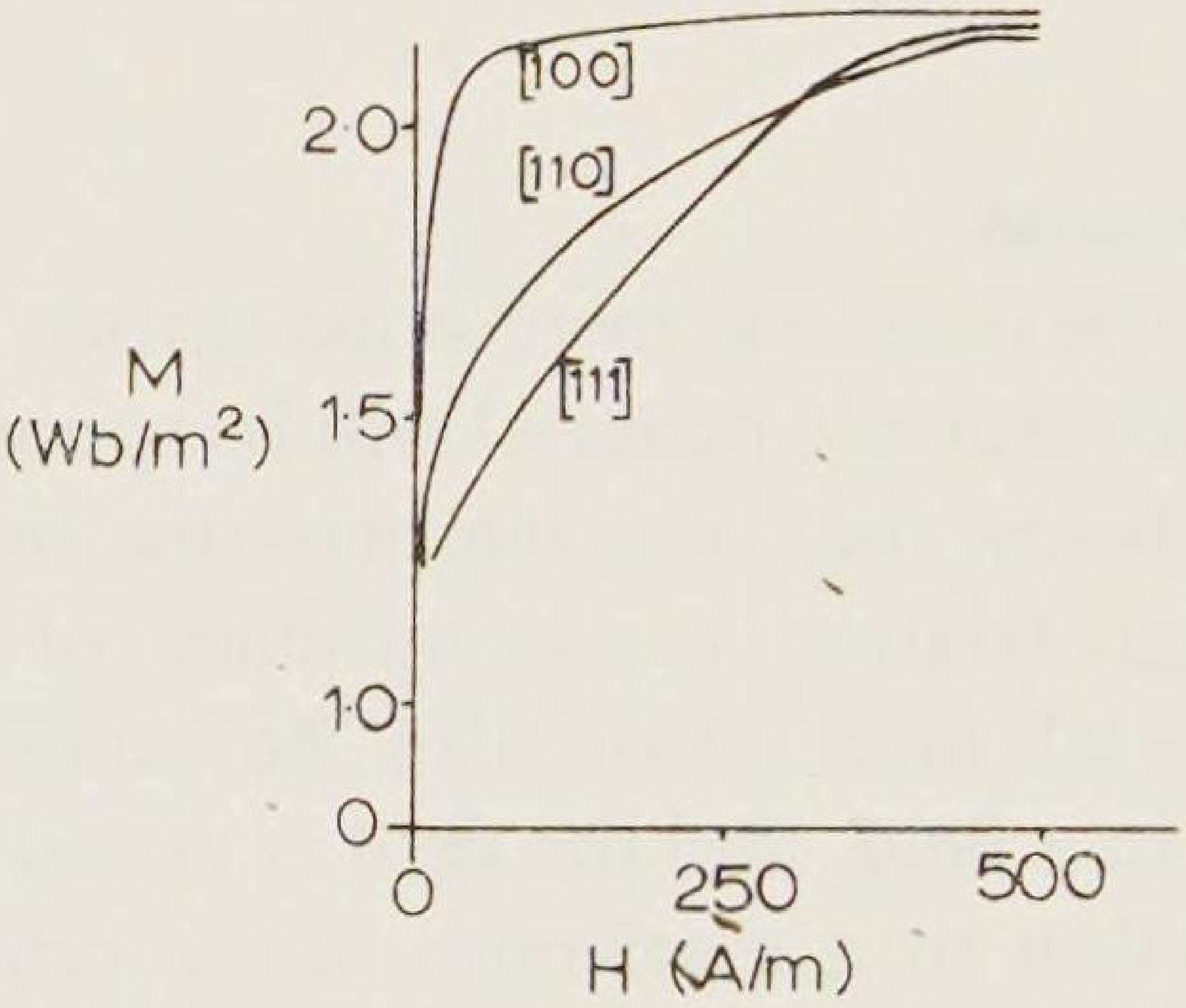


FIG. 3.5

Magnetization curves for a single-crystal of iron are shown in Fig. 3.5, each curve being labelled with the crystallographic direction to which it corresponds. It will be seen that it is much easier to magnetize the crystal in the $\langle 100 \rangle$ direction than in either of the other principal directions. Thus the $\langle 100 \rangle$ direction is called the 'easy' axis of magnetization and the phenomenon is described as magnetocrystalline anisotropy. Evidently it is energetically more favourable for the magnetization to lie in the easy direction, and the anisotropy energy involved in turning it out of this direction is discussed in greater detail in Chapter 5. For the moment it is sufficient to note the existence of this anisotropy of magnetization. We may now return to the Weiss model of magnetic domains. In Chapter 1 the idea of demagnetizing field was discussed, and if a sample were a single domain, i.e. spontaneously magnetized to saturation throughout its volume, it would have a demagnetizing field given by $N_D/\mu_0 M_s$. Now the energy associated with a magnetic field is given by equation (1.56) as

$$\frac{1}{2}\cdot\frac{N_D M_s^2}{\mu_0} = U$$

The quantity U is called the magnetostatic energy and depends on the shape of the specimen, since this determines the demagnetizing factor N_D. As an example, consider a rectangular specimen of single-crystal, having a single 'easy' direction of magnetization, which is also a single domain, as shown in Fig. 3.6(a). If this were to be converted into two domains, magnetized in opposite directions as in Fig. 3.6(b), it is obvious that a certain amount of flux closure is possible between the ends of the domains. This corresponds to an effective reduction of the demagnetizing field and, in fact, the demagnetizing factor for parallel strips of equal width is proportional to the width D of each strip and the magnetostatic energy is given by $5{\cdot}4\times10^4 M_s^2 D$ (Chikazumi, 1964) [14]. Thus in going from configurations (a) to (b) the magnetostatic energy will be halved, and for N strips as in Fig. 3.6(c), it is reduced to approximately $1/N$th of the single domain value. Thus the formation of domains reduces the magnetostatic energy considerably and, on the above arguments, it should be expected that the crystal would continue to subdivide into smaller and smaller domains indefinitely. However, as will be shown in Chapter 6, the boundaries between domains—called domain walls—required a finite energy to form them. Thus subdivision will stop when more energy is required to form the necessary area of new wall than is saved by reduction of the magnetostatic energy.

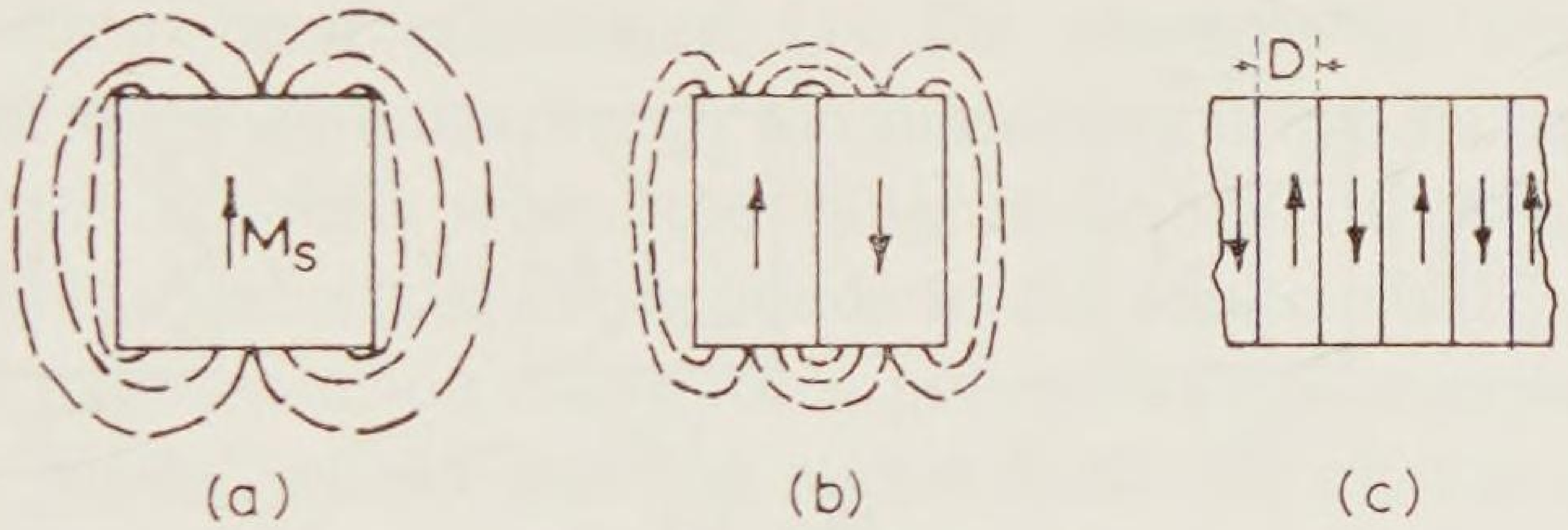


FIG. 3.6

Of course, in a configuration like that of Fig. 3.6(c), the net magnetic moment of the whole specimen will be zero, accounting for the existence of apparently unmagnetized material.

There are clearly other possible domain configurations. If we assumed the crystal to be cubic, having the ⟨100⟩-type directions as easy axes, the domain arrangement shown in Fig. 3.7(a) would correspond to complete flux closure and zero magnetostatic energy. Whether or not such a pattern would form depends on the wall energies involved. Very complicated domain patterns have been shown to exist and a fuller discussion of the factors involved will be found in Chapter 6.

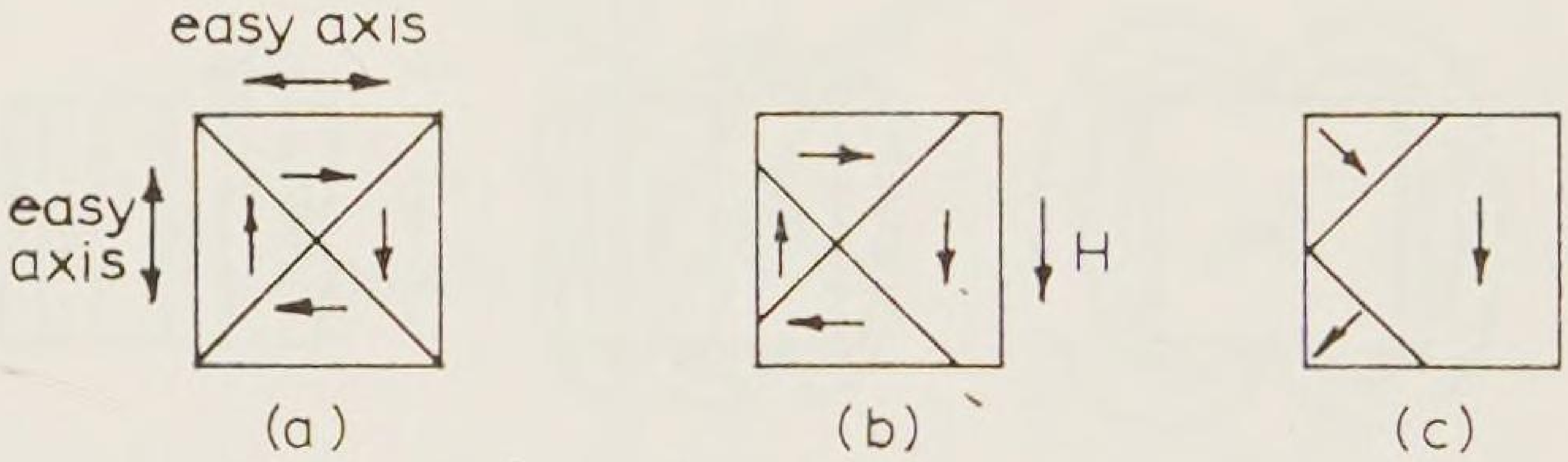


FIG. 3.7

The Magnetization Process

The process of magnetizing a specimen to saturation will evidently consist of converting it to a single domain, in which the magnetization has a direction parallel to the applied field. This may happen in two ways; domains oriented favourably with respect to the applied field may grow, by displacement of the domain walls, at the expense of those unfavourably oriented, or the magnetization within each domain may be rotated into parallelism with the applied field so that the domain walls disappear. In fact it may be shown experimentally that both these processes occur. Furthermore, domain wall motion is the process which requires least magnetic energy to actuate it and is the first to occur as the applied field is increased from zero. At small applied fields the domain walls move through small distances and return to their original positions on removal of the field; these are termed reversible displacements and correspond to the initial curved part of the magnetization curve. For larger applied fields the domain walls are found to jump discontinuously through larger distances and do not return to their original position on removal of the field. This behaviour indicates the existence of some form of barrier to domain wall movement, which the domain wall can overcome when it has acquired sufficient magnetic energy, in the field, to 'climb over' it. Once past the barrier the wall can 'slide down' on the other side. On such a model it will require a similar field to be applied in the opposite direction to move the wall back over the barrier again, thus hysteresis effects will be observed. The region of irreversible wall movement corresponds to the long centre portion of the initial magnetization curve Wall movement for the case of the cubic single crystal in Fig. 3.7(a) is shown schematically in Fig. 3.7(b). When all the unfavourable domains have been eliminated (as for example in Fig. 3.7(c)) rotation of the magnetization in the remaining domains begins and saturation is reached when the magnetizations reach alignment with the field. This stage corresponds to the top, curved portion of the initial magnetization curve for a polycrystalline sample.

Such a model would account successfully for all the observed aspects of technical magnetization such as hysteresis, coercive field and remanent flux density. We may assume that, in a polycrystalline sample, crystallite boundaries could act as obstacles, having a variety of barrier heights, to wall movement. Thus there will be no single threshold field for the onset of irreversible motion or for domain rotation and the magnetization curve will have an S-shape whilst the hysteresis loop has a rounded appearance, as in Fig. 3.4(a). If we have a single crystal of high perfection, however, it may be expected that there would be no obstacles to wall movement. Supposing the specimen to have a single easy axis and a strip-domain configuration, as in Fig. 3.6(c), we may predict that, for quite a small field, all the walls will suddenly move and completely eliminate the antiparallel domains. The hysteresis loop would then be rectangular, as in Fig. 3.4(b). If the field were applied at (say) 90° to the easy axis, however, there would be no reason for one type of domain to grow rather than another. Magnetization would then have to proceed by rotation only, gradually coming into alignment as the field was increased. This would lead to the type of loop shown in Fig. 3.4(c) in which the small hysteresis may be ascribed to small obstacles hindering rotation of the magnetization.

Experimental Techniques for Domain Observation

(1) *Bitter Patterns.* Bitter (1931) [15] and von Hamos and Thiessen (1932) [16] both independently showed that very fine particles of a ferromagnetic material (usually magnetite), suspended in a liquid on the surface being studied, tend to congregate in the region of domain walls. This is a result of the very strong local magnetic fields in the vicinity of the boundary between domains which are magnetized in differerent directions. The technique was improved by the use of a colloidal suspension of magnetite by Elmore (1937, 1938) [17] and the preparation of the surface to be examined by electropolishing to remove strained surface layers. The Elmore technique was further developed by Williams (1947) [18] and his experimental procedure, as well as his recipe for magnetite colloid, has been given by Kittel (1956) [19] who also gives the theory of the technique.

It may be assumed that the colloidal particles are in thermal equilibrium on the surface, so that their density distribution may be described by the Boltzmann equation. Thus if the density in the absence of a field is $\rho(0)$, then when a field, **H**, is present,

$$\rho(\mathbf{H}) = \rho(0)\exp(E/kT)$$

where E is the potential energy of the particle in the field. If the particle has a moment **m**, the potential energy is $\mathbf{m}.\mathbf{H}$ so that

$$\rho(H, \theta) = \rho(0)\exp\left(\frac{mH\cos\theta}{kT}\right).$$

This, of course, neglects interactions between particles.

Averaging over all possible angles:

$$\rho(H) = \frac{1}{4\pi}\int_0^\pi \exp\left(\frac{mH\cos\theta}{kT}\right)2\pi\sin\theta\, d\theta$$

$$= \frac{\sinh x}{x}$$

where $x = mH/kT$. Plotting $\rho(H)$ as a function of x it is found that the curve rises sharply for $x>3$, so that a noticeable increase in particle density due to the field will occur for $mH>3kT$. For magnetite the moment per unit volume is of the order of 0·5 weber/m^2 so that, at room temperature, the above condition will be fulfilled for fields of $\sim$0·025 amp/metre (3×10^{-4} oersted). Thus the technique is able to show up very small local variations of field in the surface under examination. Studying the domain walls of (say) Fig. 3.6(c) it is clear that the demagnetizing field will reverse direction between the adjacent domains, so that the effective field across the boundary wall will be of the order of the demagnetizing field. For a domain of width D, the field is given by $5{\cdot}4 \times 10^4 M_s D$ and assuming $D = 10^{-4}$ m and $M_s = 0{\cdot}5$ weber/m^2 it is $\sim$2·7 amp/metre, i.e. about two orders of magnitude larger than the minimum deduced above.

Even in the case of a flux-closure configuration, such as that illustrated in Fig. 3.7(a), the direction of magnetization must rotate in the plane of the domain wall. Thus, where the wall intersects the surface, there will be a local demagnetizing field arising from the component of magnetization out of the plane of the surface.

The Bitter technique has been extensively used in the study of domains and domain behaviour. The patterns obtained are usually viewed in a microscope with dark field illumination and, with a good colloid, good resolutions can be obtained. However, the colloid pattern itself gives no information on the actual magnetization directions within the domains. This difficulty can be overcome in some cases by the ‘scratch technique’. If the magnetization lies in the plane of the specimen and the surface is scratched the resulting indentation will give rise to a stray (demagnetizing) field in its vicinity if the magnetization direction crosses the indentation. The result will be a collection of colloid in the vicinity of the scratch. If, however, the magnetization is parallel to it there will be no stray field and no disturbance of the colloid pattern. In this way the axis of the magnetization can be determined.

An alternative technique for domain observation is use of the Kerr effect.

Kerr Magneto-optic Effect. If polarized light is reflected from a surface, the degree of rotation of the plane of polarization depends on the magnetic state of the surface. Thus by viewing the reflected beam through an analyser, the degree of rotation and hence the magnetization can be observed. In practice the surface is highly polished and illuminated by a powerful, parallel beam of light which falls on it through a polarizer such as a Nicol prism. The reflected beam is passed through another Nicol prism as analyser and collected in a suitable telescope. For a magnetic specimen the brightness is seen to vary across the field of view in accordance with the domain pattern. Best contrast is usually obtained when the polarizer and analyser are in the crossed position, such that one type of domain is dark. Differently oriented domains then have differing brightness according to their orientation with respect to the dark domains. By observing the direction of rotation of the plane of polarization on reflection, the direction of magnetization in the surface can be deduced.

The method has the additional advantages that it can be used over a wider range of temperatures than can the colloid technique and will reveal domain structures in cases where the colloid technique fails. It has been extensively used in studies of magnetic Ni-Fe thin films used in computer stores. (See, for instance, Wieder and Collins, 1963 [20]).

(3) *Lorentz Microscopy.* A relatively new method of domain observation makes use of the electron microscope and depends on the fact that a moving electron experiences a Lorentz force in a magnetic field, which deflects it from its original path. This force is given by the vector product $\mathbf{v} \times \mu\mathbf{M}$ when the electron beam passes through a magnetized specimen so that the deviation of the electron beam is proportional to and in a direction orthogonal to the magnetization. The emergent beams from the specimen form an image which can be brought to the focus of an electron lens and projected on to the viewing screen. The simple case of parallel 180° domains is illustrated in Fig. 3.8 and it will be seen that alternate domain walls will appear as bright and dark bands, if the lens is correctly focused on the image plane. The technique has been used successfully with thin magnetic films and has the advantage of giving high resolution so that the details of domain walls and the effects of small inhomogeneities on the magnetizations within domains can be seen. The

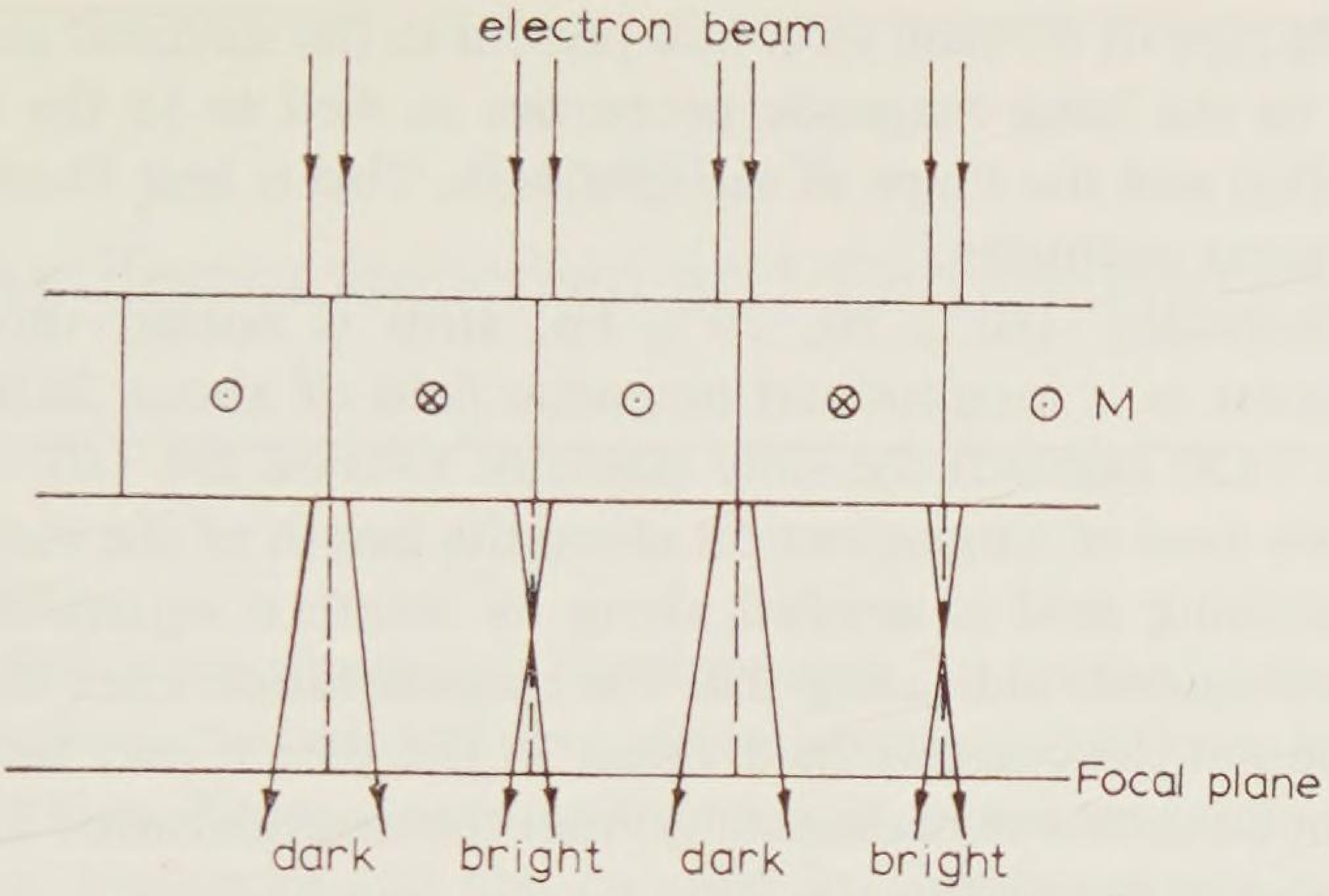


FIG. 3.8

disadvantage of the transmission technique is that it requires very thin (~1000 Å thickness) specimens. A review of the technique and its applications has been given by Cohen [21].

Magnetization Processes

The mechanism of magnetization, described earlier in this chapter, in which reversible domain wall motion is followed by irreversible motion and finally by coherent rotation of the magnetization within the domains applies to materials normally described as 'soft' ferromagnetics. These are characterized by relatively low values of the product B_rH_c and fairly small area hysteresis loops; soft iron is the archetype of such materials. For fairly 'hard' magnetic materials, having large B_rH_c values, wall displacement and rotation take place almost simultaneously. The very hardest of materials, which make the best permanent magnets, may exhibit almost no wall motion at all, magnetization being mainly by irreversible rotation.

Magnetically 'hard' behaviour is found where the barriers to wall motion are of considerable 'height' (i.e. energy). Such barriers occur by, for example, precipitating particles in an alloy and modern permanent alloy magnets such as 'Ticonal' rely on this mechanism. Further discussion will be found in Chapter 9.

The actual mechanism of the magnetization process depends also upon the *type* of domain structure present in the material and this is related to the basic magnetic properties as well as to the mode of production and the shape of the specimen. This is best illustrated by taking some examples.

If 'Permalloy' (80% Ni, 20% Fe) strip is cooled through its Curie point in a longitudinal magnetic field of about 20,000 amp/metre (~130 oersted) the only domains formed are 180° domains with their axes of magnetization along the length of the strip. When a magnetizing field is applied along its length a square hysteresis loop is obtained, indicating that the magnetization rises directly to saturation at the coercive field value. In this case it may be deduced (and can be shown experimentally) that the magnetization is entirely by motion of the 180° walls.

The case of cold-rolled 50% Ni-Fe alloy ('Isoperm') is in sharp contrast. This material exhibits an easy axis of magnetization perpendicular to the direction of cold-rolling, which is referred to as strain-induced anisotropy. If a magnetizing field is applied in the rolling direction the magnetization curve is simply a straight line with a constant slope from zero to saturation magnetization, no hysteresis being exhibited, so that the magnetization is entirely reversible. This corresponds to magnetization by reversible coherent rotation only, there being no wall motion at all. Between these two extremes virtually all possible combinations can be found. The principal factors affecting domain distribution and magnetic behaviour may be listed as follows:

(1) magnetocrystalline anisotropy, with determines the 'natural' easy axes of the crystallites;
(2) induced anisotropy, produced by strain or magnetic annealing (as for 'Permalloy') which produces an easy axis over-riding the magnetocrystalline contribution;
(3) shape anisotropy, in which the easy axis is determined by minimization of magnetostatic energy (this applies to small particles);
(4) size and orientation of the crystallites composing the specimens.

The application of these factors in practical materials is discussed in Chapter 9.

CHAPTER 4

Antiferromagnetism and Ferrimagnetics

Introduction

One of the more important developments in magnetic materials, especially from the point of view of electronic applications, has been that of ferrites. These are ceramic materials, based on oxides of iron, which are insulators and can be used in high-frequency magnetic circuits without significant eddy-current losses. They are distinguished by relatively low saturation flux densities but can be made with high relative permeabilities and with square hysteresis loops. This latter feature is of particular importance in computer applications, toroids of square-loop ferrite materials being widely used as storage elements in high-speed computer stores. A discussion of the various ferrite materials, and their principal features, is given in Chapter 8.

Development of the ferrites stems mainly from the careful experimental work of Snoek [21] in the late 1940's, together with the theoretical work of Néel [22] based on the earlier work of Van Vleck [23] who put forward the first theory of antiferromagnetism. It is with the theory that we deal first.

Antiferromagnetism

We have seen that a ferromagnetic material is distinguished by a large, positive exchange energy between neighbouring dipoles, such that the lowest energy state is that in which the dipoles are aligned with their magnetic moments parallel. In the antiferromagnetic material the exchange energy is large and negative so that the lowest-energy state is that in which neighbouring dipoles have antiparallel alignment. This means that the material would not show any spontaneous magnetization.

The actual spin arrangement can be deduced from neutron diffraction experiments. The neutron, having no charge, is not affected by electrostatic charge on ions in the lattice, but since it has a magnetic moment it can be scattered by the electron spins. By using a beam of polarized neutrons, i.e. with their magnetic moments all lying in one direction, and studying their diffraction by the crystal lattice, the spin directions of the atoms from which they are scattered can be deduced. This technique was pioneered by Shull and Smart [24] in 1949 and has since been applied to a wide range of materials. A possible antiferromagnetic arrangement of spins in a body-centred cubic lattice is shown in Fig. 4.1. In connexion with this it should be remembered that the corner atoms of the cube add up to one atom per cell so that, in an extended lattice, there are as many body-centred as there are corner atoms and the net magnetic moment can be zero. This arrangement of atoms can be regarded as consisting of two interpenetrating sub-lattices in either one of which the spins are aligned parallel, but the two sub-lattices having opposite spin

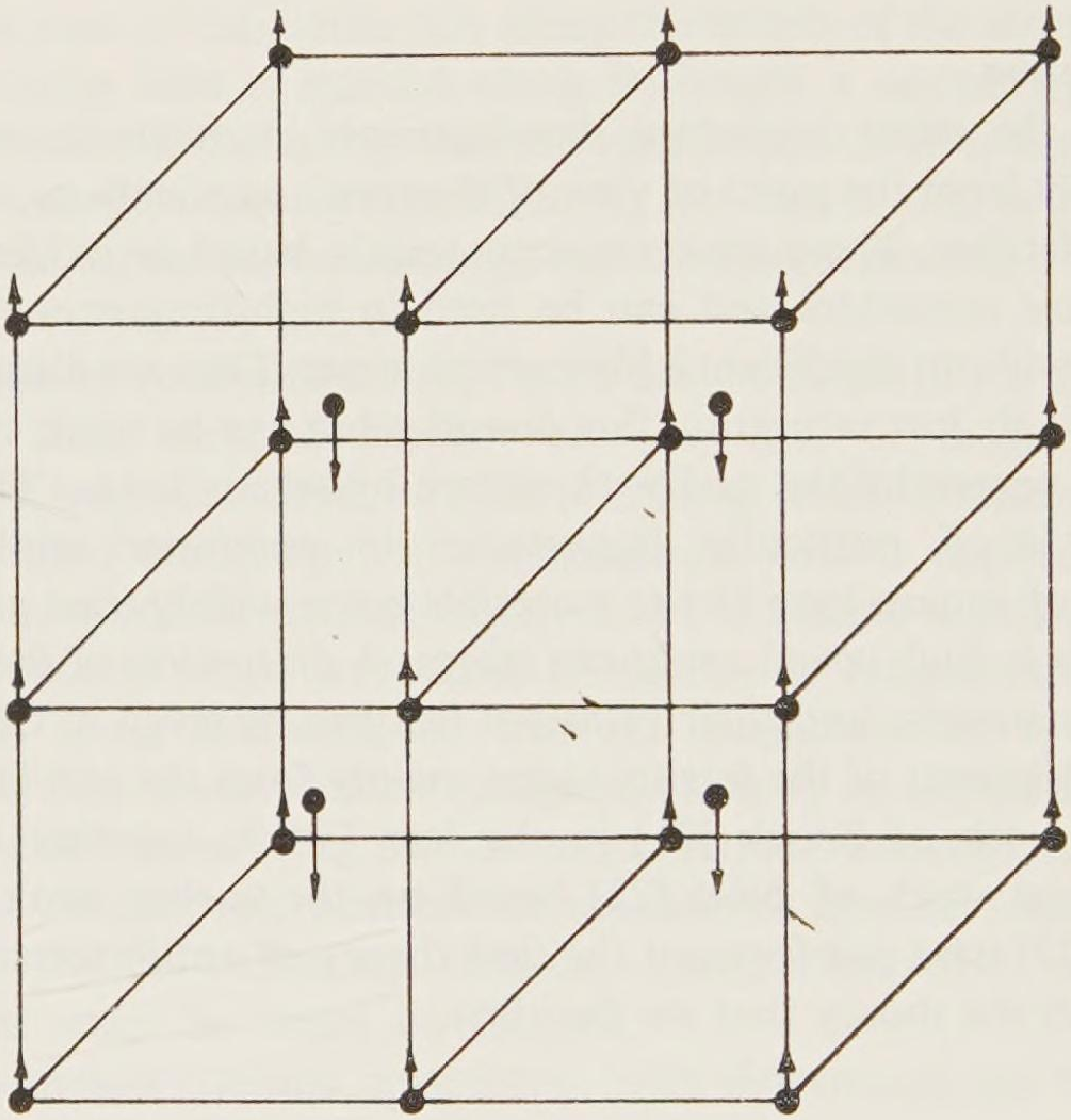


FIG. 4.1

orientation. In Fig. 4.1 the corner atoms form one lattice and the body-centred atoms form the other. The division into two sub-lattices is the basis of Néel's theory. Denoting them as A and B we assume that a spin in lattice A would be subject to the combination of an applied field H and a field due to the magnetization, M_B, of sub-lattice B. This assumes that the antiferromagnetic exchange interaction between atoms on lattice A and those on lattice B is so large that all other interactions may be neglected compared with it. Another way of putting it is that we are considering only nearest-neighbour interactions. A more refined model is dealt with later. Similarly a spin in lattice B is subject to H and M_A. The effective fields will therefore be given as

acting on A $$H_A = H - \lambda M_B \tag{4.1}$$

and acting on B $$H_B = H - \lambda M_A \tag{4.2}$$

where λ is a Weiss molecular field constant and the negative sign arises from the negative (antiferromagnetic) exchange interaction.

Following the same treatment as was used for ferromagnetism, we would expect paramagnetic behaviour for each sub-lattice at high temperature and each should obey Curie's law with the effective field replacing the applied field. Thus

$$M_A = \frac{1}{2}\frac{\mu_0 C H_A}{T} \tag{4.3}$$

and $$M_B = \frac{1}{2}\frac{\mu_0 C H_B}{T} \tag{4.4}$$

where C is the Curie constant per unit volume and the factor $\frac{1}{2}$ appears to account for the fact that each sub-lattice contains half the total number of dipoles. The total magnetization M will be given by the sum of M_A and M_B and using eqns. (4.1) and (4.2) in (4.3) and (4.4) we have

$$M = M_A + M_B = \frac{1}{2}\frac{\mu_0 C}{T}(2H - \lambda(M_A + M_B))$$

$$= \frac{\mu_0 C}{T}\left(H - \frac{\lambda}{2}M\right)$$

Thus susceptibility is given by

$$\chi = \frac{M}{\mu_0 H} = \frac{C}{T+\frac{1}{2}\mu_0\lambda C} = \frac{C}{T+\theta_N} \tag{4.5}$$

where θ_N has just half the value of the Weiss constant of equation (3.9), and is of the opposite sign.

At low temperatures the spins in any one lattice are aligned parallel so that, in considering the temperature-dependence of magnetization, we require to use the full Brillouin function, given in equation (2.23), for the magnetization. Thus we may write

$$M_A = \tfrac{1}{2}ngS\beta . B_z(y_A) \tag{4.6}$$

and

$$M_B = \tfrac{1}{2}ngS\beta . B_z(y_B) \tag{4.7}$$

where the dipoles are assumed to be due to spin, S, only and y is the argument of the Brillouin function, i.e.

$$y_A = \frac{g\beta S}{kT} H_A \tag{4.8}$$

and

$$y_B = \frac{g\beta S}{kT} H_B. \tag{4.9}$$

If we assume each sub-lattice is spontaneously magnetized to the same value, M_0, in the absence of an applied field, and that the net magnetizations of the two are antiparallel, then

$$H_A = -\lambda M_B = -\lambda(-M_0) = +\lambda M_0 \tag{4.10}$$

and

$$H_B = -\lambda M_A = -\lambda(M_0). \tag{4.11}$$

Combining eqns. (4.10) with (4.8) and (4.11) with (4.9)

$$\left.\begin{aligned} y_A &= \frac{+\lambda g\beta S M_0}{kT} \\ \text{and}\quad y_B &= \frac{-\lambda g\beta S M_0}{kT} \end{aligned}\right\} \tag{4.12}$$

Also, from (4.6) and (4.7) we have

$$M_A = M_B = M_0 = \tfrac{1}{2}ngS\beta B_z(y) \tag{4.13}$$

The value of M_0 for a given temperature may be found by plotting eqns. (4.12) and (4.13), the points of intersection giving the roots. This is shown in Fig. 4.2. From the diagram it is seen that the stable solution corresponds to the points A, A′ for which $M_A = -M_B = M_0$. If the temperature is raised the slope of the line AA′ increases so that the intersections move to BB′. If the temperature is raised sufficiently the line will become tangential to the Brillouin function at the origin, corresponding to $M_0 = 0$. This is the temperature of transition to the paramagnetic state and is called the Néel temperature, T_N.

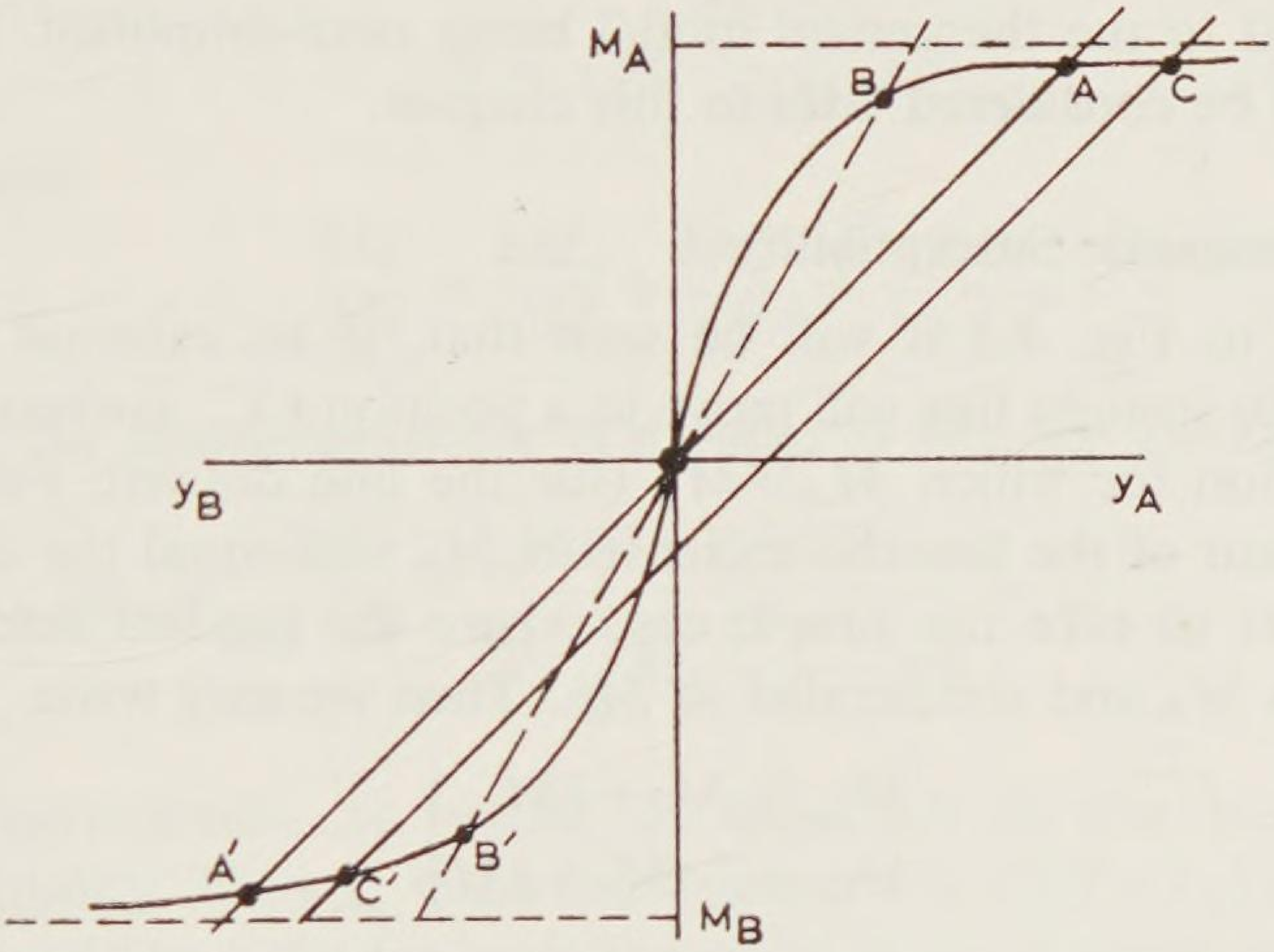


FIG. 4.2

In the vicinity of this temperature the argument, y, of B_z becomes small and we may use the approximation

$$B_z = \left(\frac{S+1}{3S}\right)y.$$

Using this in (4.12) and (4.13) we have

$$M_0 = \tfrac{1}{2}ngS\beta . \left(\frac{S+1}{3S}\right)\frac{\lambda g\beta_S M_0}{kT_N},$$

from which

$$T_N = \tfrac{1}{2}\lambda n g^2\beta^2 \frac{S(S+1)}{3k} = \tfrac{1}{2}\mu_0\lambda C = \theta_N \tag{4.14}$$

so that the constant θ_N in the Curie-Weiss equation is identified with the transition temperature T_N, as was the case with ferromagnetics. As also with ferromagnetics, practical materials may show a considerable discrepancy between θ_N and T_N, and this can be ascribed to the theoretical model being over-simplified. Refinements will be considered later in this chapter.

Antiferromagnetic Susceptibility

Referring to Fig. 4.2 it will be seen that, if an external field is applied, the straight line will move to a position CC′, corresponding to a solution for which $M_A > M_B$ (for the line drawn). For small displacement of the line the increase in M_A will equal the decrease in M_B. Let us take the simple case where the applied field, H, is parallel to M_A and antiparallel to M_B. Then we may write

$$\left.\begin{aligned} M_A &= M_0 + \delta M \\ \text{and} \qquad M_B &= -M_0 + \delta M \end{aligned}\right\} \tag{4.15}$$

and the resulting net magnetization will be $2\delta M$, its value depending on the slope of the Brillouin curve in the vicinity of the points C, C′. In the region of $T = 0$ the Brillouin function has zero slope so that δM will be zero. Thus the susceptibility, $\chi_{\parallel}$, for the case of **H** parallel to **M** will be zero at $T = 0$. For temperatures near to T_N, the Brillouin function can be approximated to

$$B_z(y') = \frac{S+1}{3S}(y') \tag{4.16}$$

where y' refers to the effective fields in the presence of an external field. Using eqns. (4.1) and (4.2) with (4.8), (4.9) and (4.16) we have

$$M_A = \tfrac{1}{2}M_s \frac{(S+1)}{3kT} g\beta(H - \lambda M_B)$$

and

$$M_B = \tfrac{1}{2}M_s \frac{(S+1)}{3kT} g\beta(H - \lambda M_A)$$

where $M_s = ng\beta S$.

Differentiating with respect to H

$$\frac{\partial M_\mathrm{A}}{\partial H} = A\left(1-\frac{\lambda \partial M_\mathrm{B}}{\partial H}\right)$$

and
$$\frac{\partial M_\mathrm{B}}{\partial H} = A\left(1-\frac{\lambda \partial M_\mathrm{B}}{\partial H}\right)$$

where
$$A = \tfrac{1}{2}M_s\frac{(S+1)}{3kT}g\beta$$

Therefore

$$\frac{\partial M}{\partial H} = \frac{\partial M_\mathrm{A}}{\partial H}+\frac{\partial M_\mathrm{B}}{\partial H} = \frac{2A}{1+\lambda A}.$$

Hence, the susceptibility for H parallel to M_A, for $T \sim T_\mathrm{N}$, is

$$\chi_\parallel = \frac{\partial M}{\mu_0\,\partial H} = \frac{M_s(S+1)g\beta/3k\mu_0}{T+\dfrac{\lambda}{2}M_s\dfrac{(S+1)}{3k}g\beta} = \frac{C}{T+\theta_\mathrm{N}}. \tag{4.17}$$

This corresponds, as would be expected, to the paramagnetic susceptibility. Thus $\chi_\parallel$ must rise from zero to $C/(T+\theta_\mathrm{N})$ at $T = T_\mathrm{N}$, so there will be a change in slope in the $1/\chi_\parallel$ vs. T curve, as shown in Fig. 4.3 at

$$T_\mathrm{N} = \frac{C}{2T_\mathrm{N}} = \frac{1}{\mu_0\lambda}. \tag{4.18}$$

For the case of the applied field perpendicular to M_A and M_B the calculation of susceptibility is very straightforward. The application of the field will cause the sub-lattice moments to turn through a small angle, ϕ, towards the field direction. The magnitude of the net moment parallel to H will be given by

$$(M_\mathrm{A}+M_\mathrm{B})\sin\phi \doteqdot 2M_\mathrm{A}\phi \tag{4.19}$$

where ϕ is small and $M_\mathrm{A} = M_\mathrm{B}$.

The turning of M_A and M_B so that they make an angle $(180° - 2\phi)$ is resisted by the Weiss field and equilibrium will obtain when the component of the Weiss field in the direction of H is just equal and opposite to H.

Thus

$$H = 2\lambda M_A \sin\phi \doteqdot 2\lambda M_A \phi \tag{4.20}$$

Therefore, using (4.19) and (4.20)

$$\chi_{\perp} = \frac{2M_A\phi}{\mu_0 H} = \frac{1}{\mu_0\lambda}. \tag{4.21}$$

The importance of this result is that it shows the susceptibility in the perpendicular direction to be constant and independent of temperature for all $T < T_N$. The predicted behaviour of both $\chi_{\parallel}$ and $\chi_{\perp}$ as temperature is varied is shown in Fig. 4.3 in which $1/\chi$ is plotted again T. The curves shown would, of course, be obtained only from single crystals; a polycrystalline sample would be expected to give a curve somewhere between the two.

In Fig. 4.4 experimental results for manganese fluoride are shown.

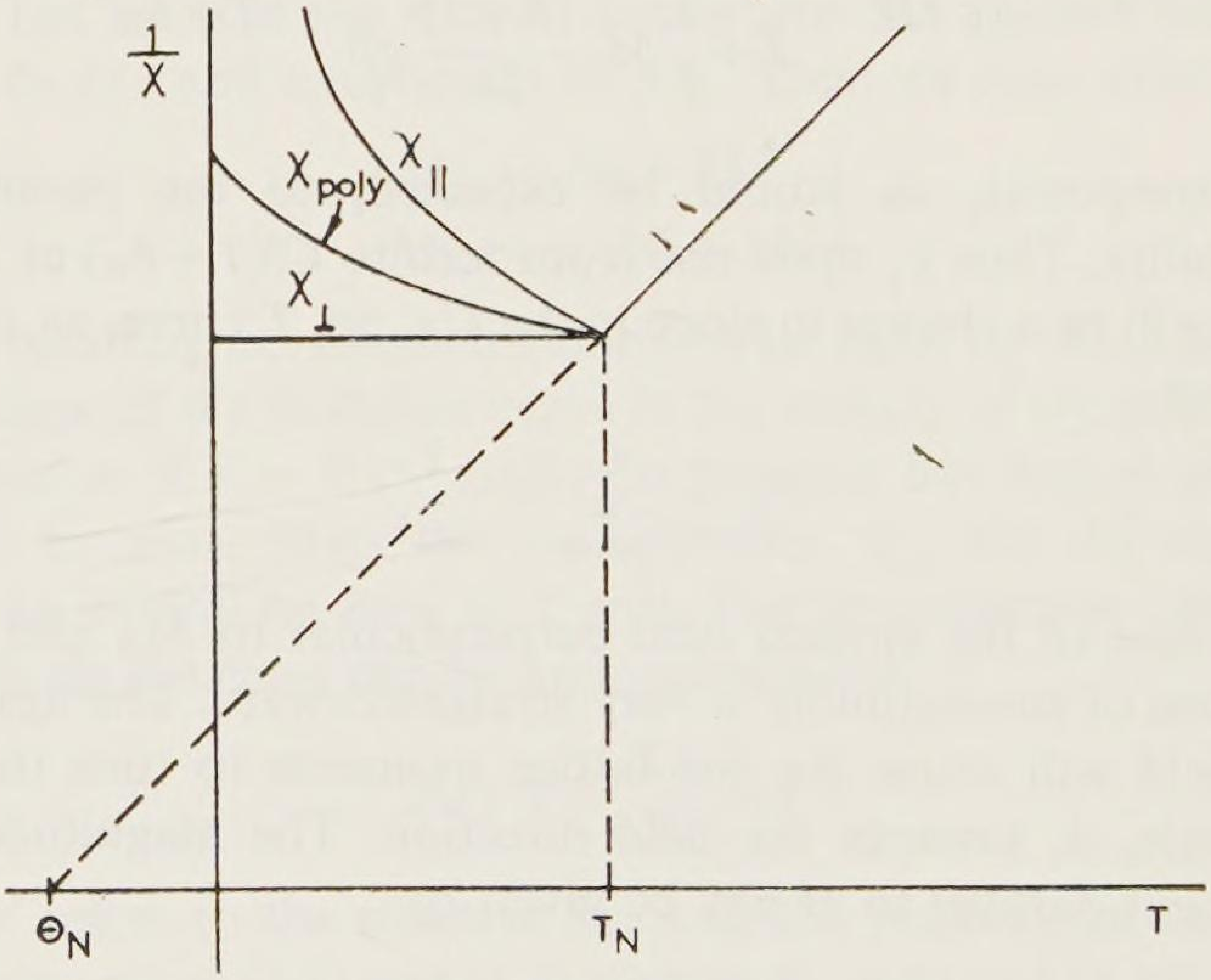


FIG. 4.3

Ferrimagnetism

A ferrimagnetic material may be defined as an antiferromagnetic in which the sub-lattice moments are unequal. This means that there will be a net spontaneous magnetization in the direction of the larger sub-lattice moment, for the case of a simple antiferromagnetic

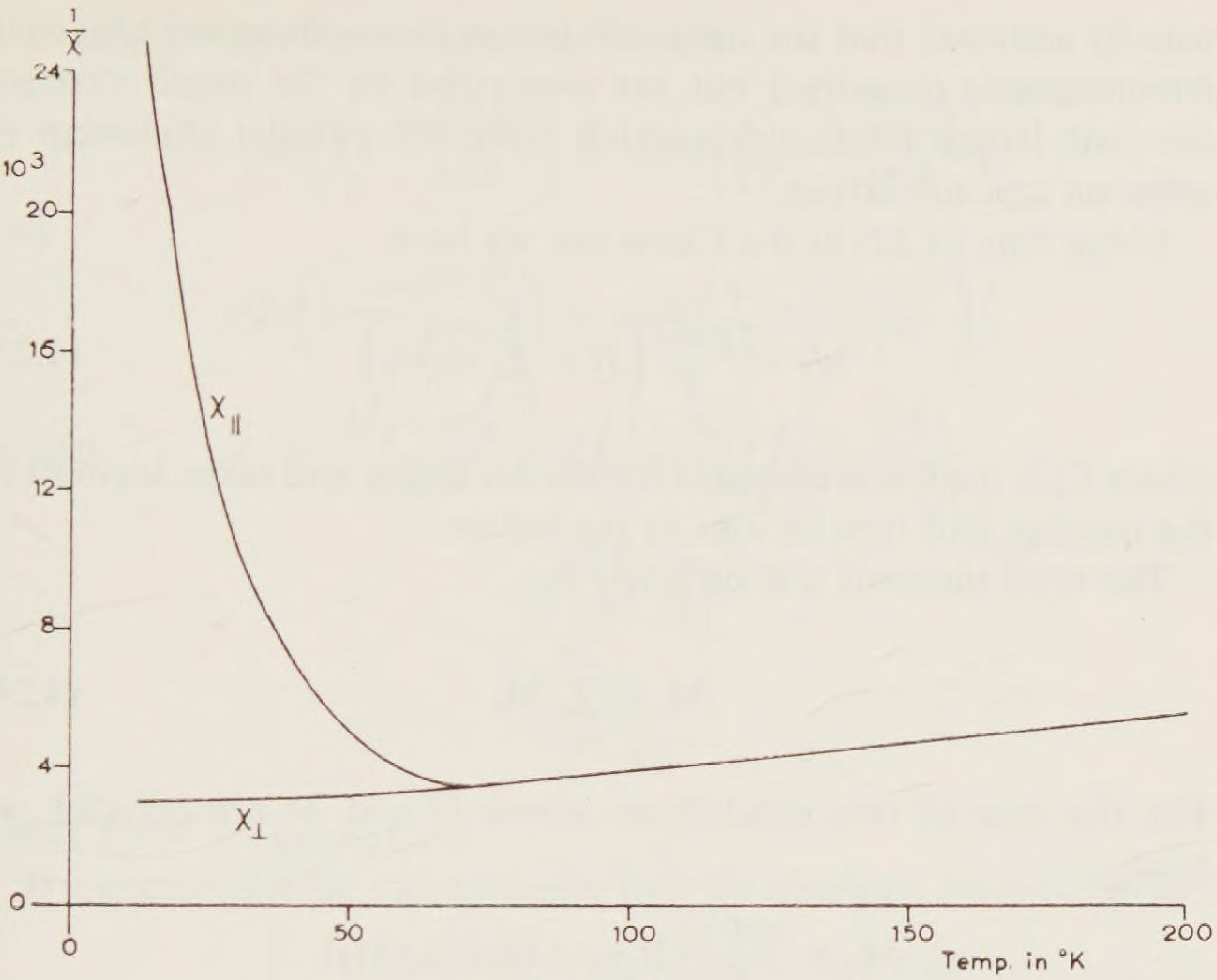


FIG. 4.4

material with strictly antiparallel sub-lattice moments. In order to deal with this theoretically we now must generalize the treatment of the previous section. The Weiss molecular field acting on a spin in the ith lattice is written, for the case of N sub-lattices as:

$$H_{wi} = \sum_{j=i}^{N} \lambda_{ij} M_j \tag{4.22}$$

where λ_{ij} is the molecular field constant between sub-lattices i and j, and is proportional to the exchange energy between them, being negative between atoms with antiparallel spins. This would include terms involving λ_{11} and λ_{22}, the *intra* sub-lattice interactions determined by the exchange forces between spins on the *same* sub-lattice, which were ignored in the simple treatment given previously. Although spins on the same sub-lattice are aligned parallel to each other this does not necessarily imply a positive exchange constant, and therefore a positive value for λ_{11}, λ_{22} etc. In ferrimagnetics it is usually assumed that the intra-sub-lattice interactions are also antiferromagnetic (negative) but are over-ruled by the much stronger *inter*-sub-lattice interactions which force the parallel alignment of spins on one sub-lattice.

Using eqn. (4.22) in the Curie law we have

$$M_i = \frac{\mu_0 C_i}{T}\left(H + \sum_{j=i}^{N} \lambda_{ij} M_j\right) \tag{4.23}$$

where C_i is the Curie constant for the *i*th lattice and takes account of the number and type of ions in the lattice.

The total moment will be given by

$$\mathbf{M} = \sum_{j=1}^{N} \mathbf{M}_j. \tag{4.24}$$

For the case of two sub-lattices where H and M are parallel, we have

$$\left.\begin{aligned} M_1 &= \frac{\mu_0 C_1}{T}(H + \lambda_{11} M_1 + \lambda_{12} M_2) \\ M_2 &= \frac{\mu_0 C_2}{T}(H + \lambda_{22} M_2 + \lambda_{12} M_1) \end{aligned}\right\} \tag{4.25}$$

(note that $\lambda_{ij} = \lambda_{ji}$).

Solving these equations simultaneously gives

$$\left.\begin{aligned} M_1\left(1 - \frac{\theta_1}{T} - \frac{(\mu_0^2 C_1 C_2 \lambda_{12}^2)/T^2}{1 - \theta_2/T}\right) &= \frac{\mu_0 H}{T}\left(C_1 + \frac{(\lambda_{12}\mu_0 C_2 C_1)/T}{1 - \theta_2/T}\right) \\ M_2\left(1 - \frac{\theta_2}{T} - \frac{(\mu_0^2 C_1 C_2 \lambda_{12}^2)/T^2}{1 - \theta_1/T}\right) &= \frac{\mu_0 H}{T}\left(C_2 + \frac{(\lambda_{12}\mu_0 C_1 C_2)/T}{1 - \theta_1/T}\right) \end{aligned}\right\} \tag{4.26}$$

where $\theta_1 = \lambda_{11}\mu_0 C_1$ and $\theta_2 = \lambda_{22}\mu_0 C_2$.

If we go back to the antiferromagnetic case treated earlier, the neglect of intra-sub-lattice interactions amounted to taking $\lambda_{11} = \lambda_{22} = 0$ so that $\theta_1 = \theta_2 = 0$. Also the sub-lattices were taken to have equal Curie constants so that $C_1 = C_2 = C/2$. Making these substitutions in the above expression leads to

$$M_1\left(1-\frac{\mu_0 C^2\lambda_{12}^2}{4T^2}\right) = \mu_0 H\left(\frac{C}{2T}+\frac{\lambda_{12}\mu_0 C^2}{4T^2}\right)$$

and

$$M_2\left(1-\frac{\mu_0 C^2\lambda_{12}^2}{4T^2}\right) = \mu_0 H\left(\frac{C}{2T}+\frac{\lambda_{12}\mu_0 C^2}{4T^2}\right)$$

so that

$$\begin{aligned}\chi_{\parallel} &= \frac{M_1+M_2}{\mu_0 H} = C\left(\frac{1}{T}-\frac{\theta_N}{T^2}\right)\Big/\left(1-\frac{\theta_N^2}{T^2}\right)\\ &= \frac{C(T-\theta_N)}{T^2-\theta_N^2}\\ &= \frac{C}{T+\theta_N}\end{aligned}$$

where $\theta_N = -\frac{1}{2}\mu_0\lambda_{12}C$.

The expression for susceptibility may be obtained from (4.26) as

$$\chi_{\parallel} = \frac{C_1(T-\theta_2(1-\beta))+C_2(T-\theta_1(1-\alpha))}{(T^2-T(\theta_1+\theta_2)+\theta_1\theta_2(1-\alpha\beta))} \tag{4.27}$$

where we take $\lambda_{11} = \alpha\lambda_{12}$ and $\lambda_{22} = \beta\lambda_{12}$.

If a net spontaneous magnetization is to appear at some Curie temperature T_c, the above expression must go to infinity at $T = T_c$. This will occur when the denominator goes to zero, so that

$$T_c^2 - T_c(\theta_1+\theta_2)+\theta_1\theta_2(1-\alpha\beta) = 0$$

i.e. when

$$T_c = \frac{\theta_1+\theta_2\pm((\theta_1-\theta_2)^2+4\theta_1\theta_2\alpha\beta)^{1/2}}{2} \tag{4.28}$$

As T becomes very large θ_1/T, θ_2/T and $\theta_1\theta_2/T^2$ become much less than unity so that we get

$$\underset{T\to\infty}{\chi(T)} \doteqdot \frac{C_1+C_2}{T}. \tag{4.29}$$

Thus if we plot $1/\chi$ vs. T it will tend to a constant slope of $1/(C_1+C_2)$ as T becomes large and will fall to zero at $T = T_c$ as shown in Fig. 4.5.

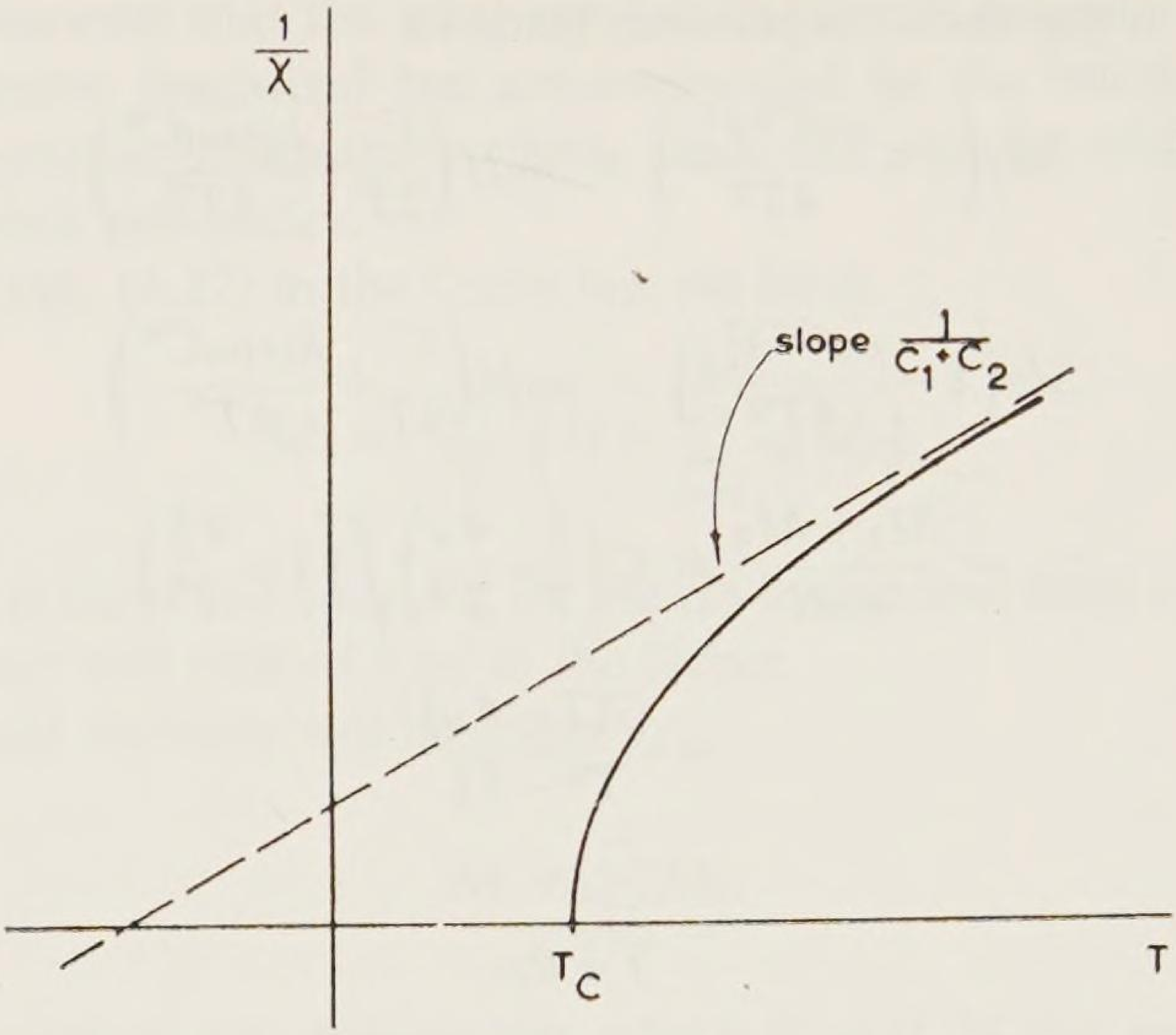


FIG. 4.5

Temperature Dependence of Magnetization

The behaviour of the susceptibility below the Curie temperature depends on the manner in which total magnetization varies with field. The total, saturation magnetization, M_s, is equal to the algebraic sum of the sub-lattice magnetizations, M_1 and M_2 in the case of two sub-lattices. Both M_1 and M_2 will vary with temperature according to the Brillouin function, i.e.

$$M_i(T) = M_{si}B_{zi}(y_i)$$

but the argument, y_i, will probably be different for each. Thus the total magnetization will not necessarily have the Brillouin form, as a function of temperature. This is shown for two possible cases in Figs. 4.6(a) and 4.6(b). Néel [25] calculated the temperature-dependence of M_s for the two sub-lattice, collinear model using suitable approximations for $B_z(y)$ and predicted the temperature-dependence for various relative values of the parameters α and β introduced in the previous section. These fit with experimental results for various compounds in the class of the ferrospinel, ferrogarnet and magnetoplumbite crystal classes. Two typical

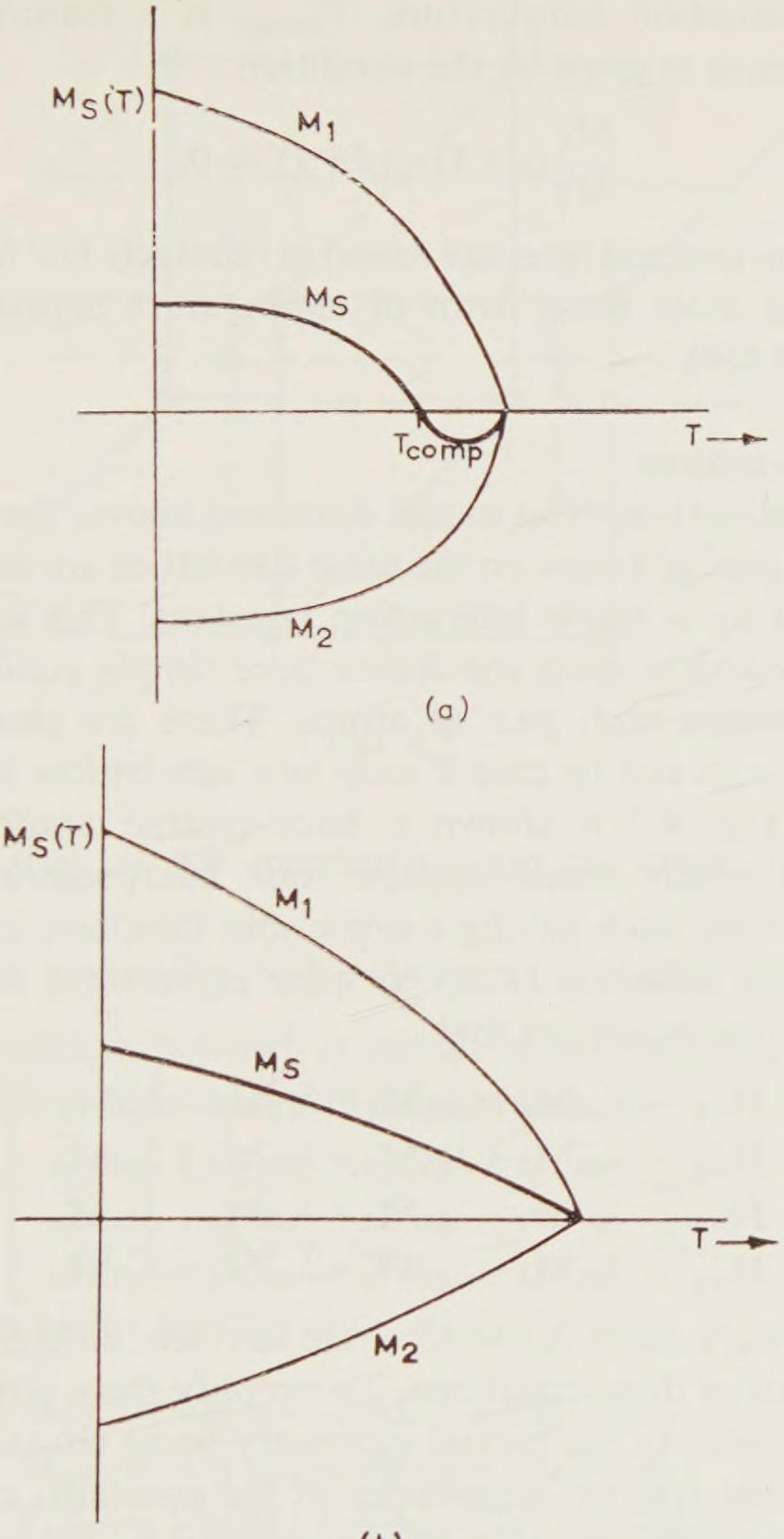


FIG. 4.6

cases are shown in Figs. 4.6(a) and (b) for garnet-type ferrimagnetic materials of the type $5Fe_2O_3 - 3M_2O_3$ where M is Gd in Fig. 4.6(a) and is Y in 4.6(b). (Note that the field applied for measurement purposes orients M_s so that the reversal of sign in Fig. 4.6(a) would not appear in practice.)

The compensation temperature, T_{comp}, is a feature unique to ferrimagnetics; it is given by the condition

$$\frac{M_2}{M_2}(\alpha+1)-(\beta+1) = 0.$$

Compensation temperatures are found in relatively few ferrimagnetic materials, the most usual form of temperature dependence being that of Fig. 4.6(b).

Multiple Sub-lattices

In the two sub-lattice, Néel model discussed above, the interactions between any pair of atoms on the same sub-lattice are assumed to be characterized by a single interaction constant. This would be the case if, for example, each sub-lattice were simple cubic with equal distances between each pair of atoms. There are clearly cases in which this would not be true if only two sub-lattices are used; for example in Fig. 4.7 is shown a body-centred antiferromagnetic arrangement which would require four interpenetrating simple-cubic sub-lattices, each having a single spin direction, to describe it. We may apply equation (4.22) to write expressions for the Weiss field in each sub-lattice, so that

$$\left.\begin{aligned}\mathbf{H}_{w1} &= \lambda_{11}\mathbf{M}_1+\lambda_{12}\mathbf{M}_2+\lambda_{13}\mathbf{M}_3+\lambda_{14}\mathbf{M}_4\\ \mathbf{H}_{w2} &= \lambda_{21}\mathbf{M}_1+\lambda_{22}\mathbf{M}_2+\lambda_{23}\mathbf{M}_3+\lambda_{24}\mathbf{M}_4\\ \mathbf{H}_{w3} &= \lambda_{31}\mathbf{M}_1+\lambda_{32}\mathbf{M}_2+\lambda_{33}\mathbf{M}_3+\lambda_{34}\mathbf{M}_4\\ \mathbf{H}_{w4} &= \lambda_{41}\mathbf{M}_1+\lambda_{42}\mathbf{M}_2+\lambda_{43}\mathbf{M}_3+\lambda_{44}\mathbf{M}_4\end{aligned}\right\} \quad (4.30)$$

Since we always have $\lambda_{ij} = \lambda_{ji}$ there are ten different molecular field constants in these equations. To simplify them any further it is necessary to refer to the crystal symmetry being considered so that estimates of the relative magnitudes of the constants can be made, based on nearest-neighbour, next-nearest-neighbour interactions and so on. A rigorous treatment of this has been given by Goodenough [26].

A special case of interest has been treated by Yafet and Kittel [27] in which a triangular spin arrangement may occur. If we take

$$\left.\begin{aligned}&\mathbf{M}_1+\mathbf{M}_2 = \mathbf{M}_\mathrm{A}\\ \text{and}\qquad &\mathbf{M}_3+\mathbf{M}_4 = \mathbf{M}_\mathrm{B}\end{aligned}\right\} \quad (4.31)$$

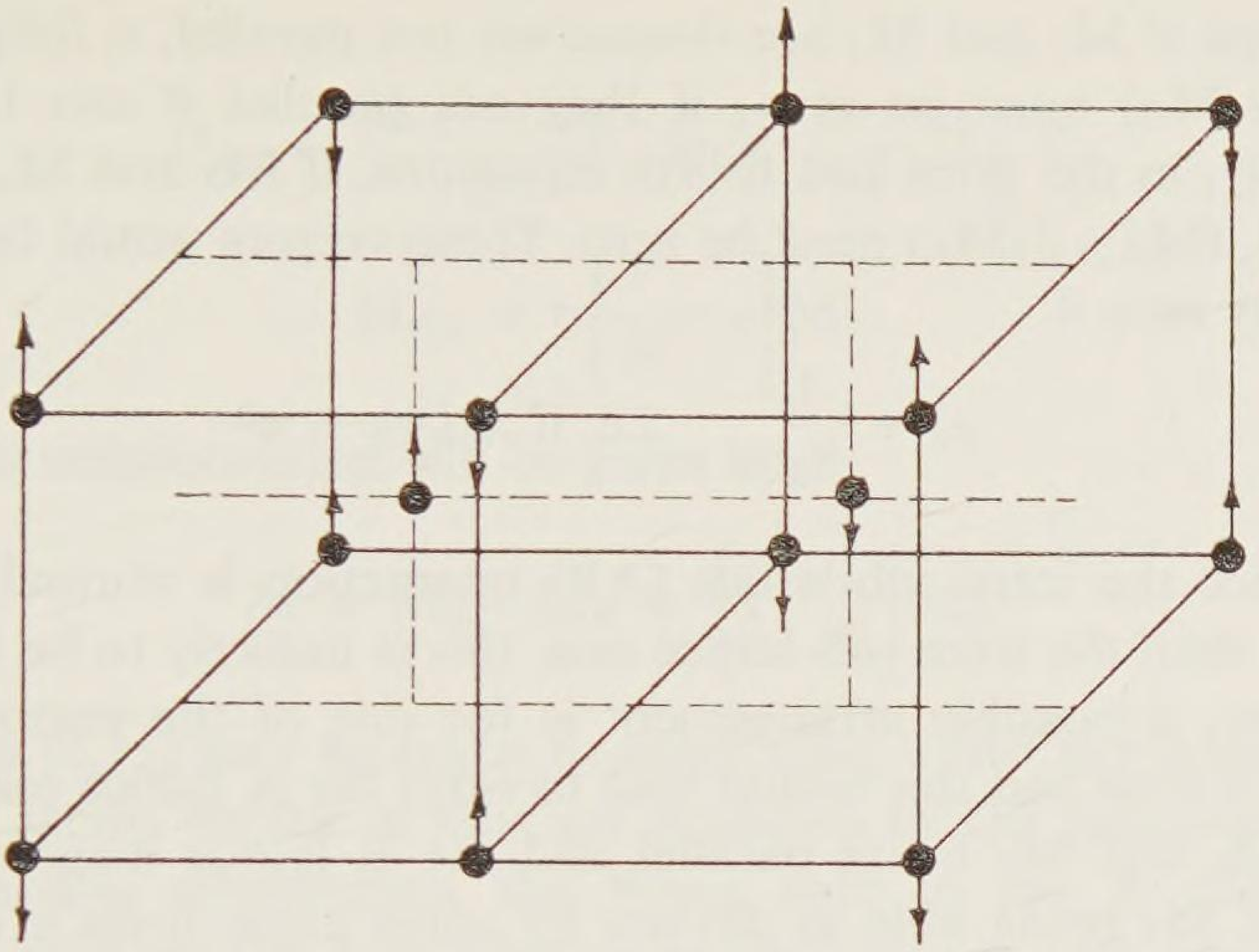


FIG. 4.7

and assume that all AB interactions are the same, then we may write

$$\lambda_{13} = \lambda_{14} = \lambda_{24} = \lambda_{23} = -n \text{ (say).}$$

(This is a condition met with in common ferrimagnetic materials). Using a similar notation to that used before, let

$$\lambda_{11} = -\alpha_1 n = \lambda_{22}, \qquad \lambda_{12} = -\alpha_2 n$$

$$\lambda_{33} = -\beta_1 n = \lambda_{44}, \qquad \lambda_{34} = -\beta_2 n.$$

then equation (4.31) may be re-written as

$$\left.\begin{aligned} \mathbf{H}_{w1} &= -n(\alpha_1-\alpha_2)\mathbf{M}_1+\alpha_2\mathbf{M}_{\mathrm{A}}+\mathbf{M}_{\mathrm{B}}) \\ \mathbf{H}_{w2} &= -n(\alpha_1-\alpha_2)\mathbf{M}_2+\alpha_2\mathbf{M}_{\mathrm{A}}+\mathbf{M}_{\mathrm{B}}) \\ \mathbf{H}_{w3} &= -n(\mathbf{M}_{\mathrm{A}}+\beta_2\mathbf{M}_{\mathrm{B}}+(\beta_1-\beta_2)\mathbf{M}_3) \\ \mathbf{H}_{w4} &= -n(\mathbf{M}_{\mathrm{A}}+\beta_2\mathbf{M}_{\mathrm{B}}+(\beta_1-\beta_2)\mathbf{M}_4) \end{aligned}\right\} \qquad (4.32)$$

Now the magnetization in a sub-lattice must necessarily lie parallel to the Weiss field in that sub-lattice, since this is the whole basis of the Weiss theory. Referring to equations (4.32), which are vector equations, from the first it follows that the vector $(\alpha_2\mathbf{M}_{\mathrm{A}}+\mathbf{M}_{\mathrm{B}})$ must be parallel to $\mathbf{M}_1$ and from the second it must be parallel to

$\mathbf{M}_2$. Thus if $\mathbf{M}_1$ and $\mathbf{M}_2$ are themselves not parallel, it follows that $(\alpha_2\mathbf{M}_A+\mathbf{M}_B)$ must be zero; if they are parallel it can be finite. Similarly, in the third and fourth equations, if $\mathbf{M}_3$ and $\mathbf{M}_4$ are not parallel, $(\mathbf{M}_A+\beta_2\mathbf{M}_B)$ must be zero. These vectors would be simultaneously zero if

$$\alpha_2 = \frac{1}{\beta_2} \qquad \text{i.e. if } \lambda_{34}\lambda_{12} = n^2$$

and since the inter sub-lattice (AB) interaction is normally much greater than the intra sub-lattice one, this is unlikely to be the case. However, a possible arrangement is for one of the vectors to be zero at a time and this would lead to (say) the A lattice magnetizations $\mathbf{M}_1$ and $\mathbf{M}_2$ being parallel and the B lattice magnetizations $\mathbf{M}_3$ and $\mathbf{M}_4$ being able to deviate by some angle from a collinear alignment as shown in Fig. 4.8.

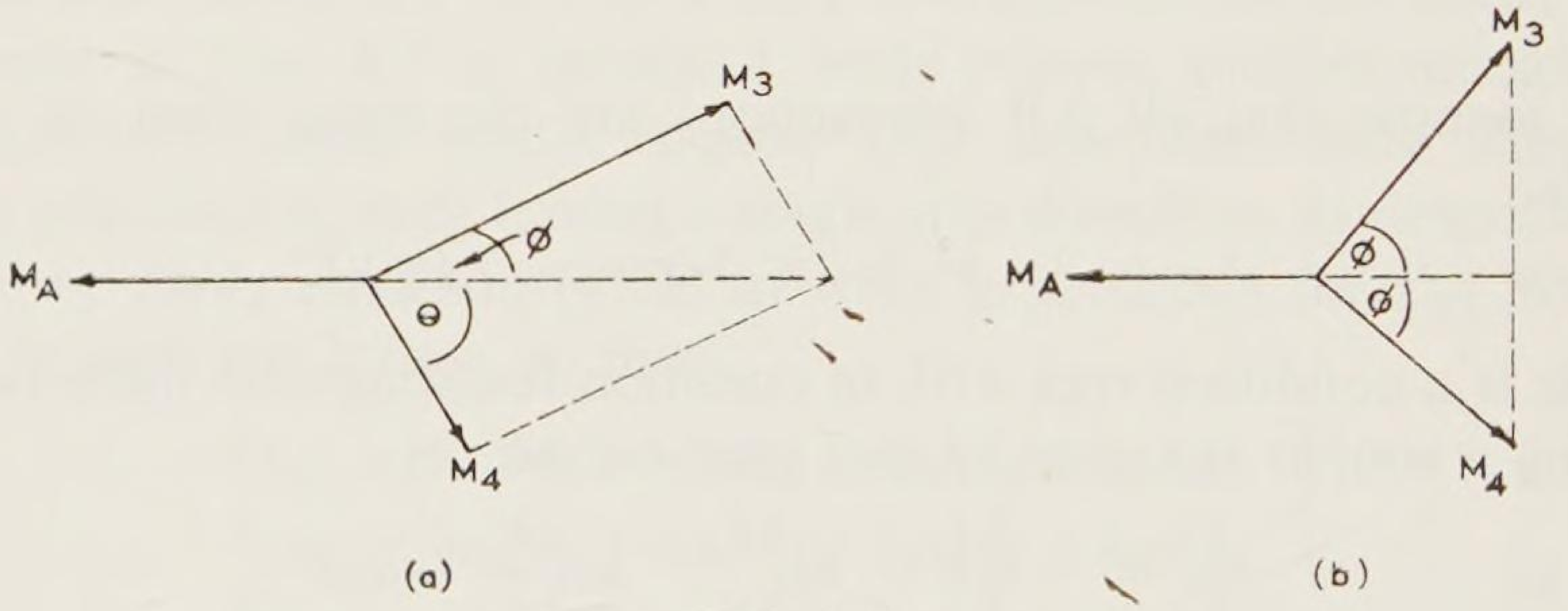


FIG. 4.8

For this case, from equations (4.32), we have

$$\left.\begin{aligned} H_{w3} &= n(\beta_2-\beta_1)M_3 \\ H_{w4} &= n(\beta_2-\beta_1)M_4 \end{aligned}\right\} \tag{4.33}$$

and

Since we have

$$(\mathbf{M}_A+\beta_2\mathbf{M}_B) = 0 \tag{4.34}$$

by substitution in (4.32) one obtains

$$H_{w1}+H_{w2} = -n\left[(\alpha_1+\alpha_2)M_A-\frac{2}{\beta_2}M_A\right].$$

Defining

$$H_{w\mathrm{A}} = \frac{H_{w1}+H_{w2}}{2} \quad \text{and} \quad \alpha = \frac{\alpha_1+\alpha_2}{2}$$

then

$$\mathbf{H}_{w\mathrm{A}} = n\left[\frac{1}{\beta_2}-\alpha\right]\mathbf{M}_{\mathrm{A}}. \tag{4.35}$$

The total magnetization will be given by

$$\mathbf{M} = \mathbf{M}_{\mathrm{A}}+\mathbf{M}_{\mathrm{B}} = \left[1-\frac{1}{\beta_2}\right]\mathbf{M}_{\mathrm{A}} \tag{4.36}$$

Equation (4.35) may be taken to mean that, whilst the interactions between atoms on the A sites are modified by the presence of the B sub-lattice, nevertheless the Weiss field in the A sub-lattice depends only on the A sub-lattice magnetization. This means that $\mathbf{M}_{\mathrm{A}}$ will have the normal temperature-dependence found in a ferromagnetic and so, from equation (4.36), the total magnetization will have a ferromagnetic-type temperature dependence.

From the condition given in equation (4.34) it is clear that $\mathbf{M}_{\mathrm{A}}$ and $\mathbf{M}_{\mathrm{B}}$ must be antiparallel; thus, referring to Fig. 4.8(a), when $\mathbf{M}_3$ is not equal to $\mathbf{M}_4$ the angles by which they deviate from the direction of M_{A} must also differ. For the (common) case of $M_3 = M_4$ the angles will also be equal as shown in Fig. 4.8(b). We then have, using (4.37), $-\beta_2(2M_3)\cos\phi = M_{\mathrm{A}}$, so that, in general,

$$\cos\phi = \frac{M_{\mathrm{A}}}{2\beta_2 M_{\mathrm{B}}} < \frac{M_{\mathrm{A}}}{\beta_2 M_{\mathrm{B}}}. \tag{4.37}$$

The minimum angle is, of course, $\phi = 0$, when $\mathbf{M}_3$ and $\mathbf{M}_4$ are parallel and collinear with $\mathbf{M}_{\mathrm{A}}$. If we imagine M_{A}, M_{B} and α_2 to be fixed and consider β_2 to vary we get three possible conditions. For $\beta_2 < M_{\mathrm{A}}/M_{\mathrm{B}}$ the condition for angle formation is not fulfilled and $\mathbf{M}_3$ and $\mathbf{M}_4$ will be parallel, both being antiparallel to $\mathbf{M}_{\mathrm{A}}$ and normal ferrimagnetism will occur. As β_2 increases, at the point where it equals $M_{\mathrm{A}}/M_{\mathrm{B}}$ angle formation can appear on the B sub-lattice, and as it continues to increase the angle ϕ becomes progressively smaller. When $\beta_2 \geqslant 1/\alpha_2$ we have seen that angle formation can occur on both A and B lattices. This, however, is an unstable state and, in fact, an antiferromagnetic state appears with M_1 antiparallel to M_2 and M_3 antiparallel to M_4.

Starting from absolute zero with $M_A > M_B$ and the ferrimagnetic arrangement shown in Fig. 4.9(a), if we assume that M_A decreases more rapidly than M_B as the temperature is raised, the above increase of β_2 will be simulated and at some temperature a triangular spin arrangement will occur as in Fig. 4.9(b). As the temperature continues to increase, an antiferromagnetic arrangement can appear, as in Fig. 4.9(c). It may be that the Curie temperature of either the A or B lattice will be reached before this stage, in which case one lattice will become paramagnetic and the other will become antiferromagnetic. This will be the Curie temperature for the material. Finally there will be a temperature at which the other sub-lattice also becomes paramagnetic and this will be the Néel temperature.

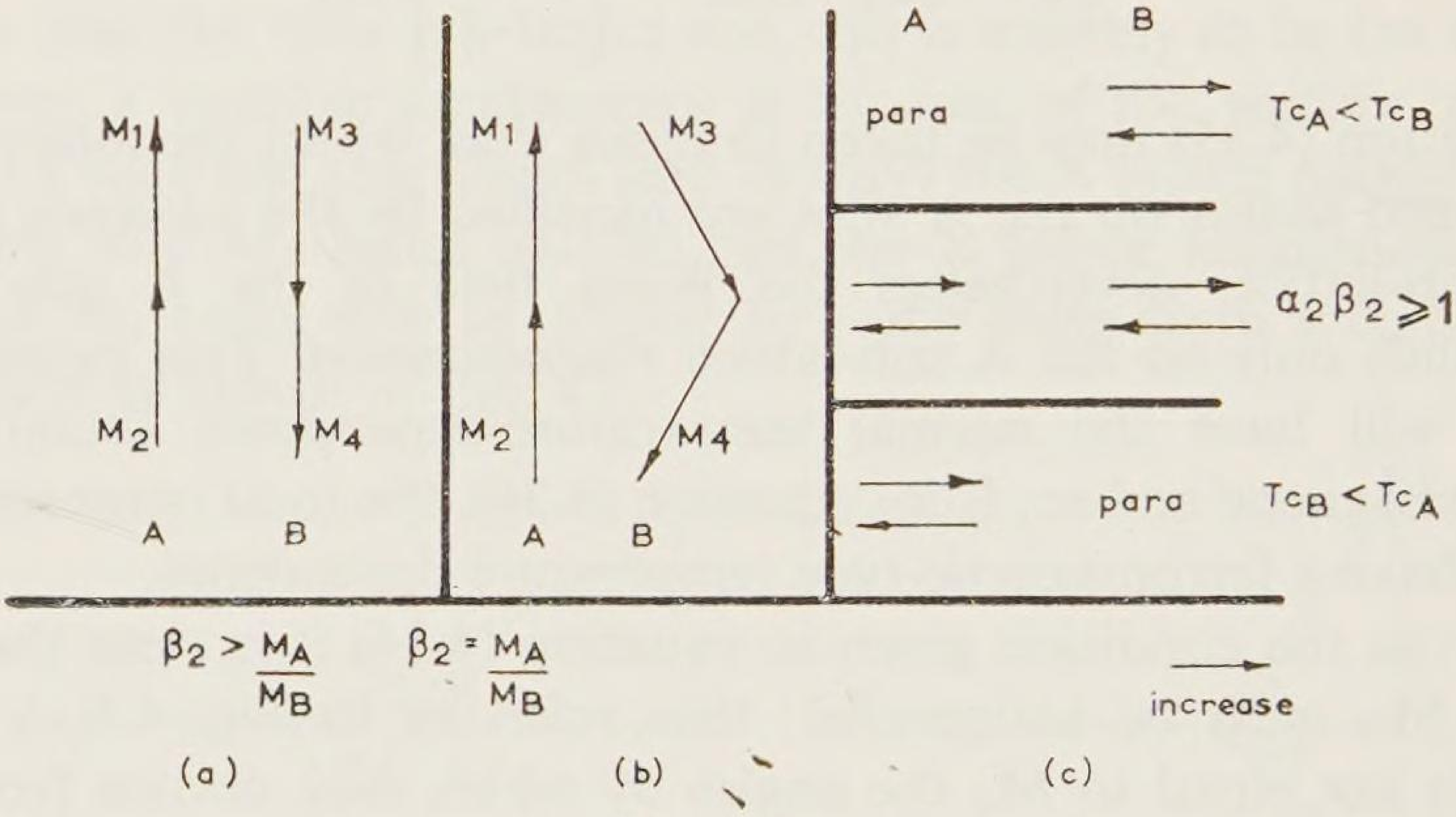


FIG. 4.9

Recent work by Dey and Anderson [28] on the cadmium manganites has provided experimental evidence for canted spins on a B sub-lattice. In Fig. 4.10 is shown $1/\chi$ as a function of temperature for $Cd_{0 \cdot 8}Mn_{2 \cdot 2}O_4$. The cadmium and Mn^{2+} ions occupy tetrahedral sites and form the A sub-lattice, whilst Mn^{3+} ions occupy the octahedrally co-ordinated B-sites in a tetragonally distorted spinel lattice.

Referring to Fig. 4.10 region I on the curve corresponds to paramagnetism. At the temperature T_N antiferromagnetic order

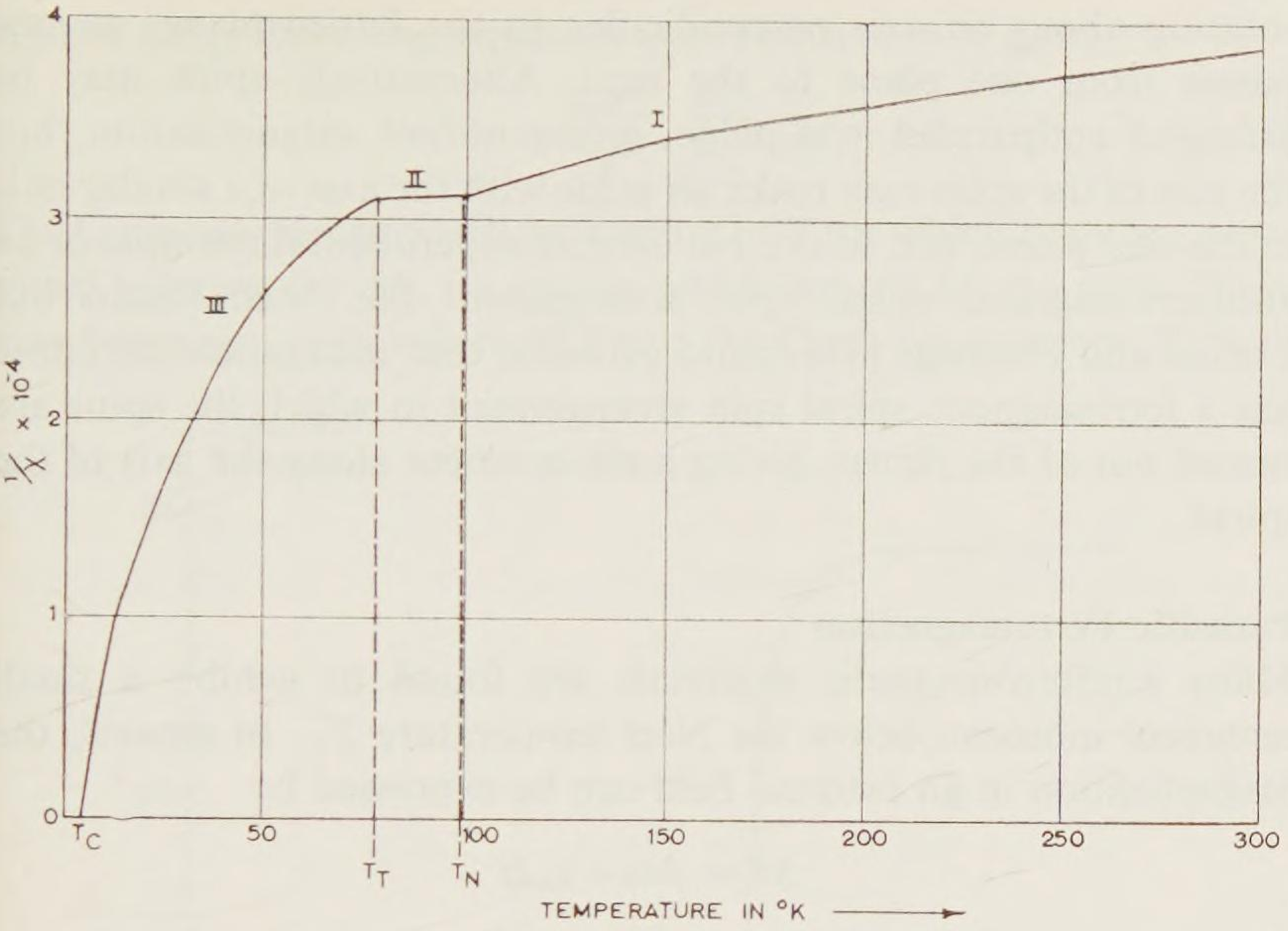


FIG. 4.10

appears on the B sites, the A sites remaining paramagnetic. The constancy of $1/\chi$ with temperature in region II arises from the fact that the susceptibility measured is $\chi_{\perp}$, which was shown, earlier in this chapter, to be temperature-independent. At the temperature T_{T} there appears a canted spin arrangement in the B lattice and this persists over region III, down to the Curie temperature when normal ferrimagnetism appears. There is a difference in this case, however, which is that, due to the low value of M_{A} arising from the presence of non-magnetic ions on the A sites, the Curie temperature of the A lattice is very low. The canted spins on the B sites are thus accompanied by paramagnetic disorder in the A sub-lattice, rather than ferromagnetic order as in the theory quoted above.

Spiral Spins

Lyons and Kaplan [29] have given a generalized treatment of spin configurations, which includes the Yafet-Kittel triangular spin arrangement, which suggests the possibility of a spiral spin arrangement. At its simplest this would correspond to the net magnetization rotating about an axis perpendicular to the lattice planes as one passes from one plane to the next. Alternatively spins may be arranged antiparallel in a plane, giving no net magnetization, but the axis of the spins may make an angle with the axis of a similar pair in the next plane, and so on. No definite experimental example of an antiferromagnetic spiral spin arrangement has been found but Corliss and Hastings [30] found evidence that manganese chromite has a ferrimagnetic spiral spin arrangement in which the spins are canted out of the planes, giving a net moment along the axis of the spiral.

Parasitic Ferromagnetism

Many antiferromagnetic materials are found to exhibit a small remanent moment, below the Néel temperature T_N. In general, the magnetization in an external field can be expressed by

$$M = M_0 + \chi_a H$$

where χ_a is the normal antiferromagnetic susceptibility and M_0 is the remanent moment. For a long time M_0 was supposed to be due to ferromagnetic impurities in the sample, but M_0 is frequently almost independent of temperature, and this does not agree with such a model. Dzialoshinski [31] pointed out that this remanent moment was strongly dependent on crystal symmetry and showed that, in many cases, it could be accounted for by assuming that spins on two antiparallel lattices are slightly canted from the true antiparallel position. If the symmetry of the crystal is such that the canting of both sub-lattices is in the same direction there will be a net remanent moment. A typical case is that of hematite (Fe_2O_3) illustrated in Fig. 4.11. The temperature T_M corresponds to a transition between a parasitic ferromagnetic state for $T > T_M$, with spins generally in the basal plane of the orthorhombic crystal, to an antiferromagnetic state for $T < T_M$, with the spins aligned along the c-axis.

Curie Temperature in Multiple Sub-lattices

The Curie temperature for a sub-lattice is defined as that temperature at which spontaneous magnetization appears in the absence of an applied field. Rewriting equation (4.23) with $H = 0$ gives

$$M_i = \frac{\mu_0 C_i}{T} \cdot \sum_{j=1}^{N} \lambda_{ij} M_j \tag{4.38}$$

The Curie temperature will be a value of T for which there is a non-trivial solution for M_i, i.e. one in which not all M_j are zero. There may be several such values of T and the Curie temperature, T_c, will

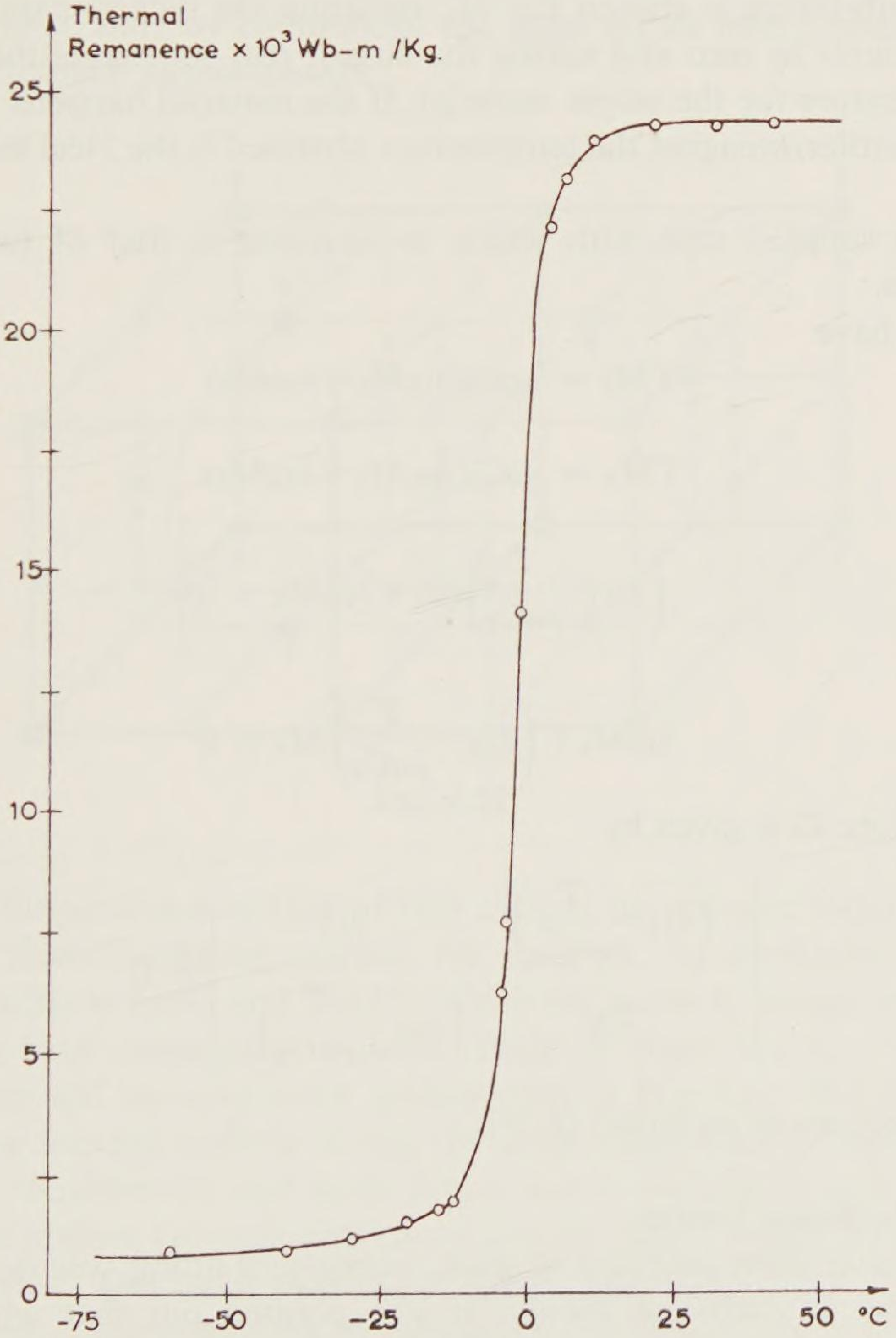


FIG. 4.11

be the highest of them and corresponds to that arrangement of spins which gives the lowest energy at temperatures below T_c. It will be found by setting the determinant of the coefficients of M_i to zero, which gives an equation of the Nth degree in T_c. Since all the sub-lattices are tied together, only one transition temperature is realized physically and this corresponds to the largest real root. Other real solutions correspond to less stable phases. Thus whichever sub-lattice is chosen for M_i, equating the determinant of the coefficients to zero and taking the largest real root gives the Curie temperature for the whole material. If the material happens to be a pure antiferromagnet the temperature obtained is the Néel temperature.

The simplest case with which to illustrate is that of two sub-lattices.

We have

$$TM_1 = \mu_0 C_1(\lambda_{11}M_1 + \lambda_{12}M_2)$$

and

$$TM_2 = \mu_0 C_2(\lambda_{21}M_1 + \lambda_{22}M_2)$$

i.e.

$$\left(\lambda_{11} - \frac{T}{\mu_0 C_1}\right)M_1 + \lambda_{12}M_2 = 0$$

and

$$\lambda_{12}M_1 + \left(\lambda_{22} - \frac{T}{\mu_0 C_2}\right)M_2 = 0$$

Therefore T_c is given by

$$\begin{vmatrix} \left(\lambda_{11} - \dfrac{T}{\mu_0 C_1}\right) & \lambda_{12} \\ \lambda_{12} & \left(\lambda_{22} - \dfrac{T}{\mu_0 C_2}\right) \end{vmatrix} = 0$$

This reduces to equation (4.28).

Superexchange Energy

In Chapter 3 the coupling of spins on adjacent atoms was described in terms of exchange energy. It was pointed out that when the exchange integral is positive the coupling is such as to maintain the spins parallel and ferromagnetism results. In antiferromagnetic materials the exchange integral is negative and therefore favours antiparallel orientation of spins on adjacent atoms. These are the simple cases where partially filled *d*-orbitals are able to overlap allowing exchange of *d* electrons between the atoms. In compounds, however, the ions with partially-filled *d*-orbitals are too far apart for much direct overlap to take place and if magnetic ordering occurs it can do so only by coupling of the spins via an intermediate ion. This is termed '*superexchange*'.

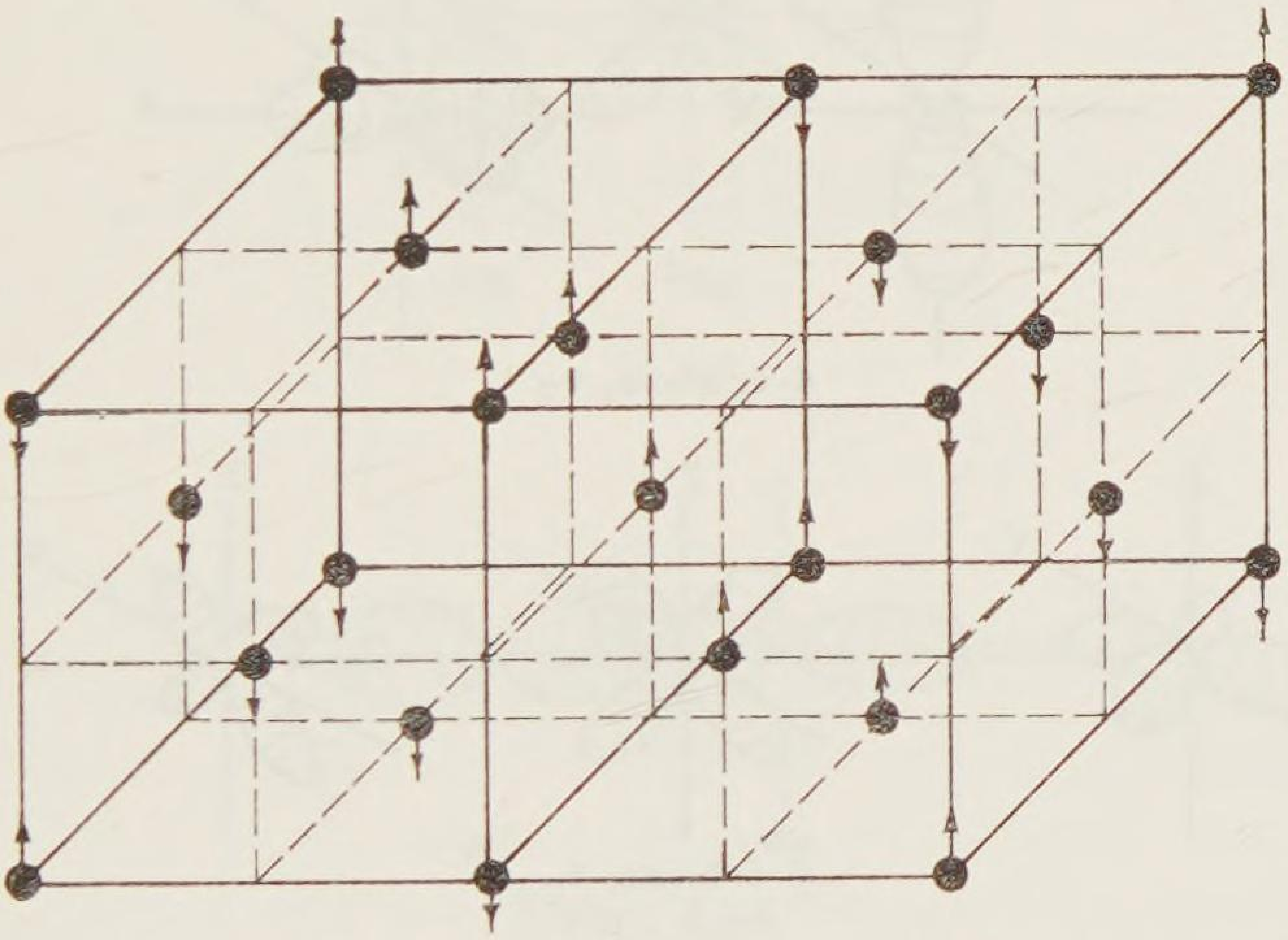
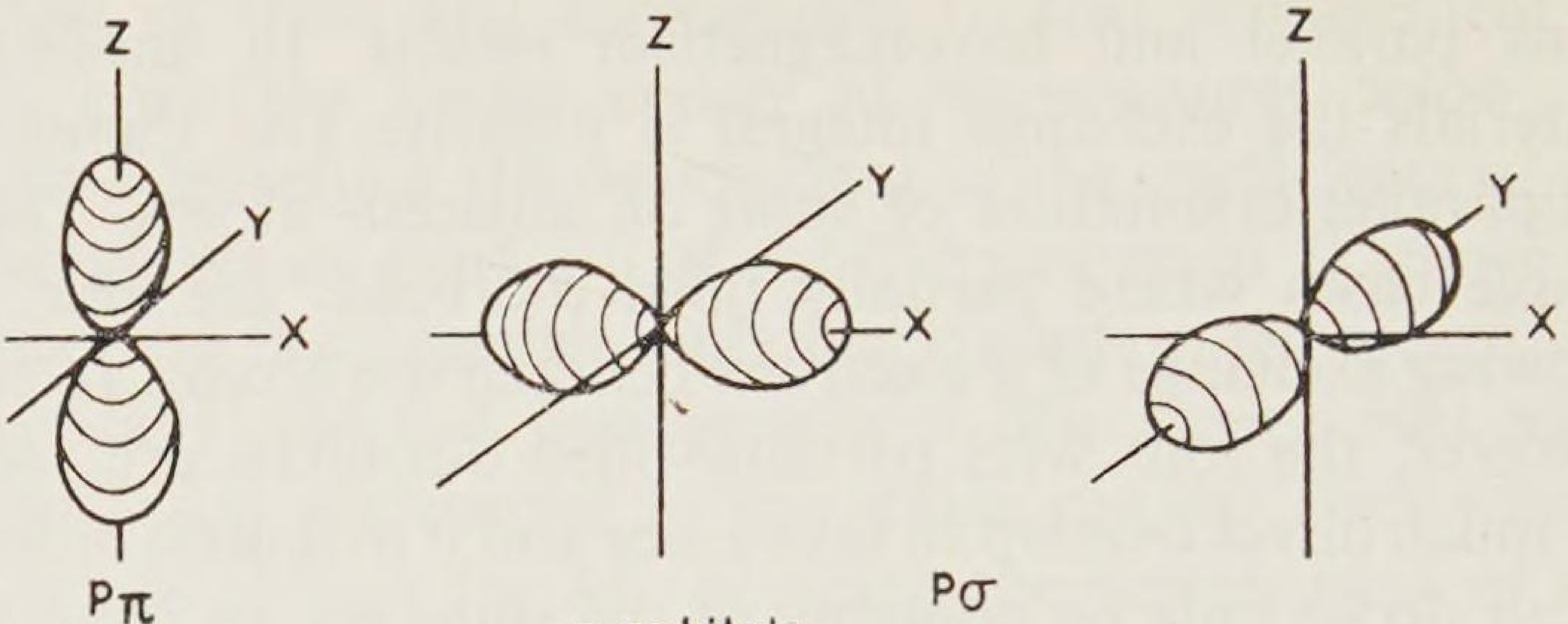
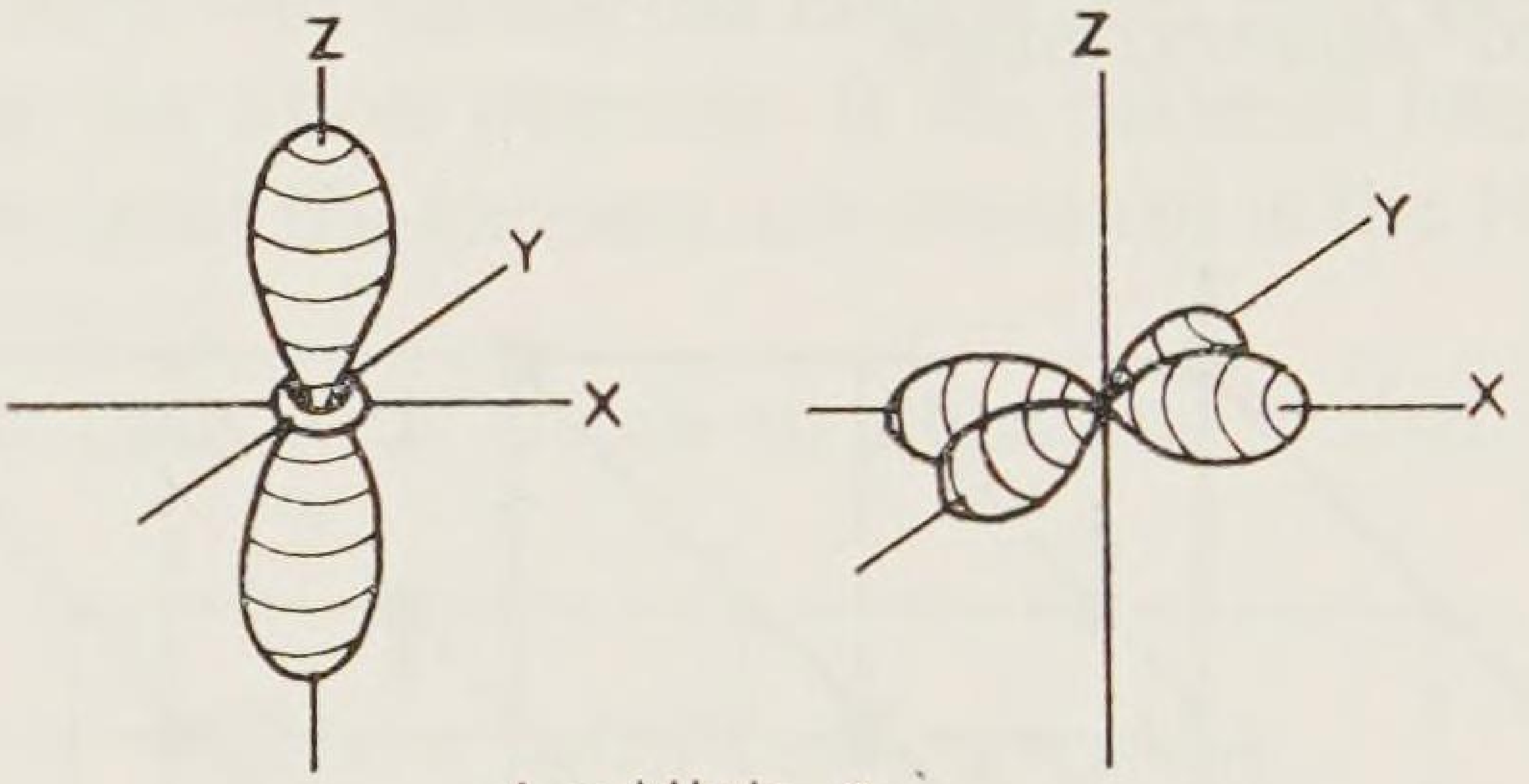


FIG. 4.12

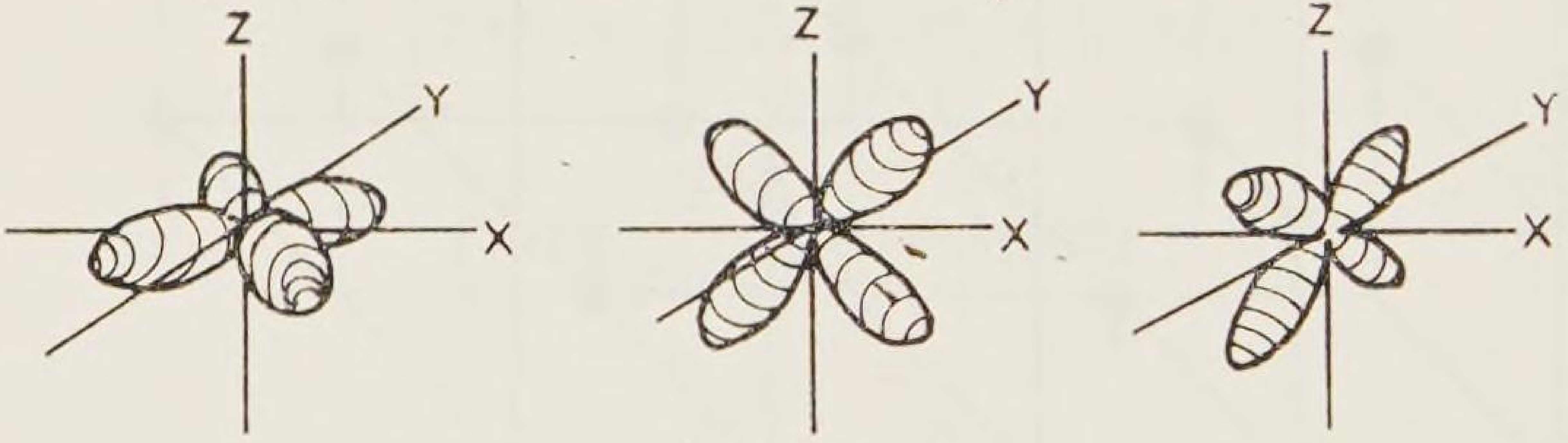
The simplest case is that of two cations on opposite sides of an anion. Such symmetry occurs, for example, in transition metal oxides such as MnO and CoO in which the anion is oxygen and the cations have incomplete *d*-shells. Each of these has a rock-salt structure and the spin-order is illustrated in Fig. 4.12. The oxygen ions are located midway along the cube edges and in the body-centred position so that each cation has a six-fold co-ordination with the anions, i.e. each metal atom has six oxygen atoms as nearest neighbours. The outer orbitals of the oxygen atom form a *p*-shell and there is the possibility of interactions arising due to overlap of the oxygen *p*-orbitals and the cation *d*-orbitals.



p-orbitals.



d-orbitals, e_g



d-orbitals, t_{2g}

FIG. 4.13

The orbitals concerned are, of course, described by wave-functions having a certain symmetry, given by the angular part of the wave function. These symmetries may be altered, in the solid state, by the crystal field so that the crystalline symmetry of the solid being considered must be taken into account. In a cubic crystal, like that described above, the *p* and *d* wave-functions have the symmetries shown in Fig. 4.13. From these diagrams it can be seen that the *p*-orbitals p_π and the *d*-orbitals t_{2g} are antisymmetric with respect to the *xy* plane, whilst the *p*-orbitals p_σ and the *d*-orbitals e_g are symmetric, about this plane. Now the antisymmetric and symmetric orbitals will always be orthogonal to each other so the net overlap between them will be zero. However p_π will overlap t_{2g} since they are both antisymmetric and p_σ will overlap e_g, both being symmetric, as shown in Fig. 4.14. It may readily be predicted from these diagrams that, since the overlap is greater in the $p_\sigma - e_g$ case, this will be the more powerful coupling. Such overlaps will lead to bonding, i.e. to the possibility of electron transfer between the atoms. The partial transfer of electrons from one atom or ion to another is equivalent to forming a covalent bond between them. When this occurs, the occupancy of the orbitals will determine the spin states of the transferred electrons and a number of possibilities may be distinguished. For the case of MnO, the cation, Mn^{2+}, has five d-electrons so that each of the five d-orbitals is singly occupied and the state of the ion is represented by $t_{2g}^3 e_g^2$. The two s-electrons of the Mn are transferred to the oxygen atom, completing the p-shell so that each of the p-orbitals is doubly occupied containing two electrons of antiparallel spin. Now if partial covalent bonding can occur simultaneously on opposite sides of the anion there will be a correlation between the spins of the electrons arising in the following way: on the cation the spins of the five electrons will all be parallel (Hund's

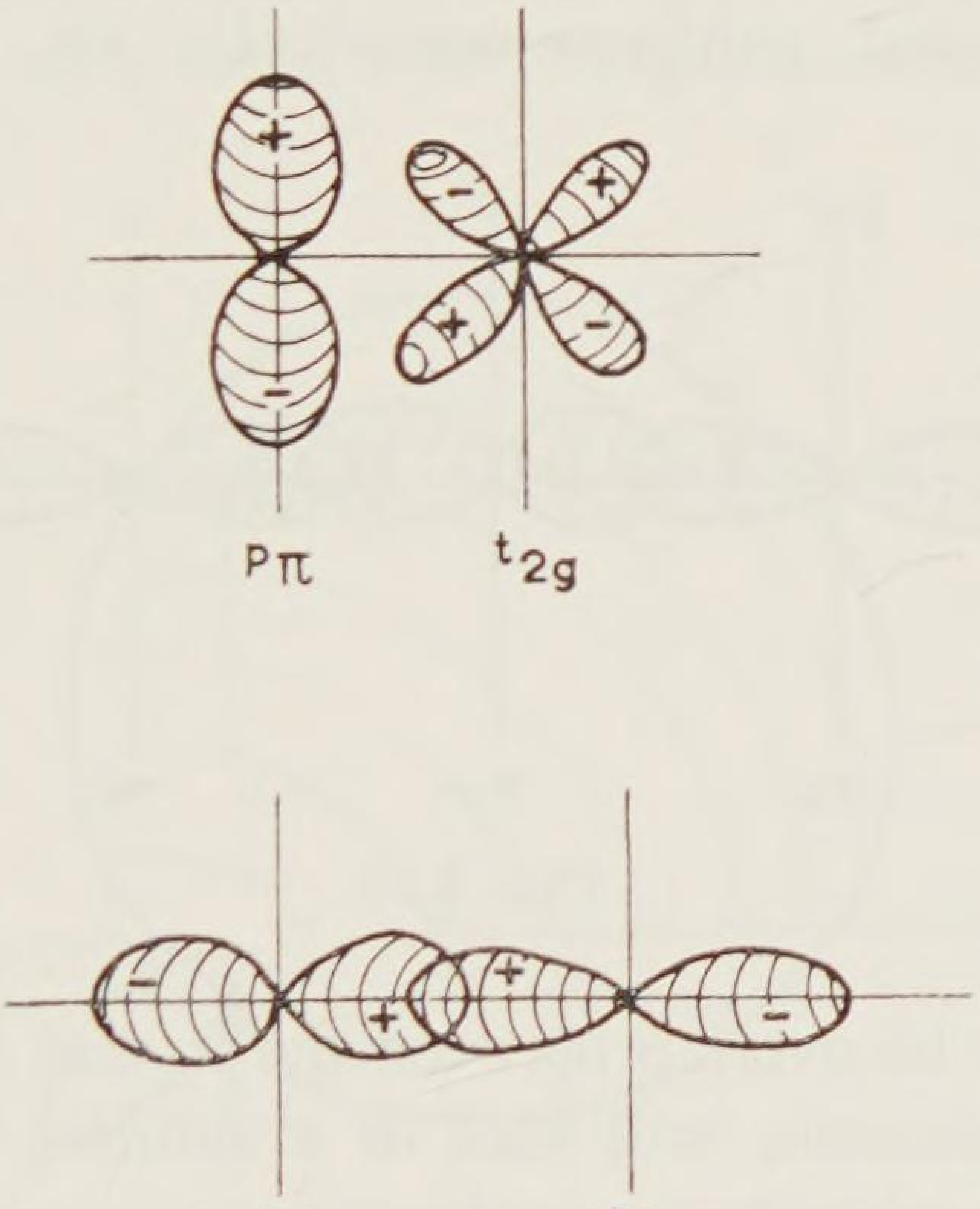


FIG. 4.14

rule) and with five orbitals available each will be singly occupied. Thus, taking the $p_\sigma - e_g$ overlap an electron may be transferred from p_σ to e_g, but, by the Pauli principle, it must have a spin opposite to that of the electron already in the e_g orbital. The same transfer taking place on the other side of the anion obeys the same rules but, since the two p_σ spins must themselves be antiparallel, the spin on the second cation must be antiparallel to the first for the transfer to take place. This situation is illustrated in Fig. 4.15. Thus the two cations are coupled antiferromagnetically via the intermediate anion.

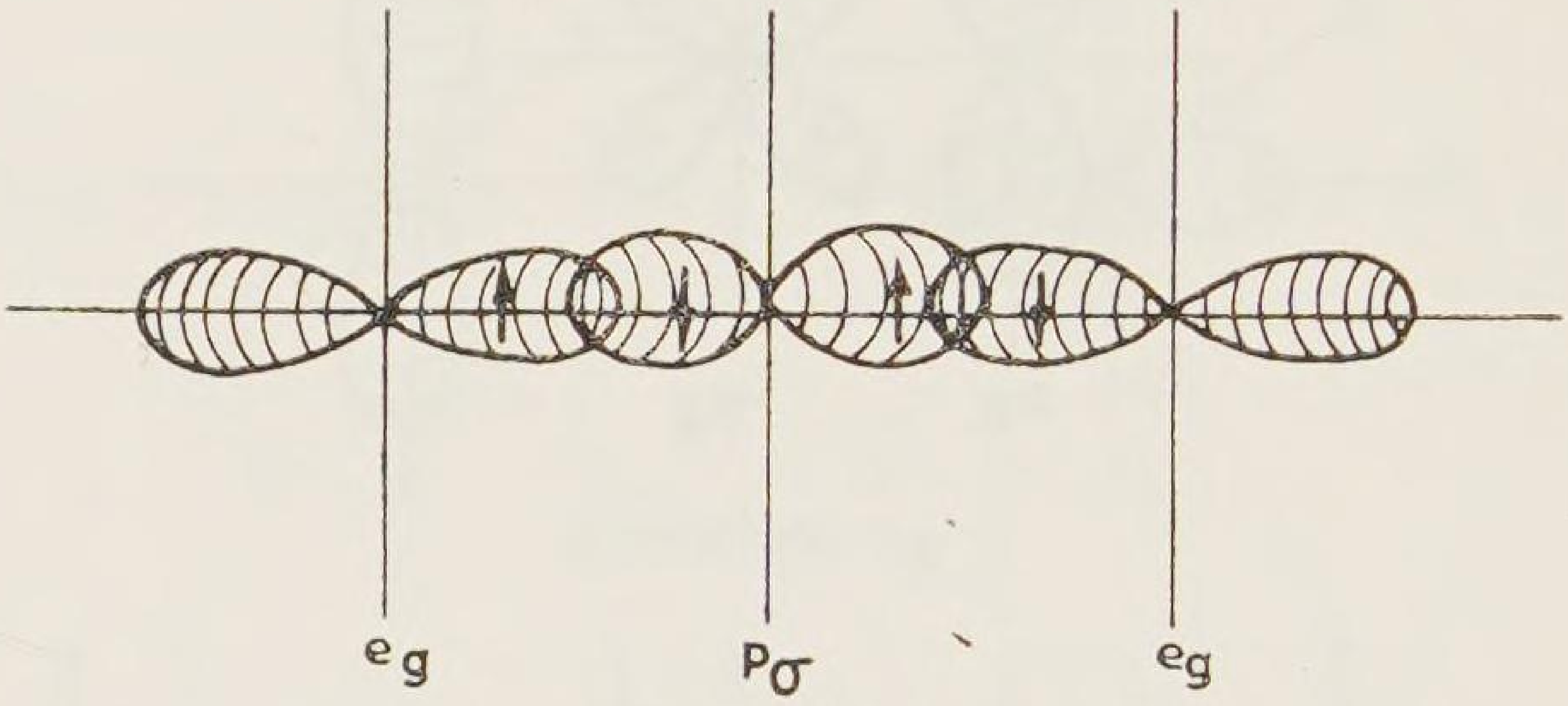


FIG. 4.15

There will also be overlap between the p_π and t_{2g} orbitals which, by the same arguments, will lead to a similar, but weaker, antiferromagnetic coupling between the cations.

For the case of cations with only three d-electrons, the t_{2g}^3 orbitals are full and the e_g orbitals empty. If there is overlap between e_g and p_σ it would seem that the p_σ electrons could be transferred with either spin into the empty e_g orbitals. However, the two p_σ electrons must have opposite spins and Hund's rule demands that all the d electrons, when there are less than five of them, must be of the same spin on one cation. Thus the cation will again be antiferromagnetically coupled.

Hypothetically the superexchange mechanism could give rise to ferromagnetic exchange. If one had a d^5 cation on one side and a d^3 on the other side of the anion and consider $p_\sigma - e_g$ overlap, then the p_σ electron which transfers to the d^5 cation must have the opposite spin to the electron in the half-filled e_g orbital. On the other side, however, the p_σ electron can transfer to the empty e_g orbital and will, by Hund's rule, have the same spin as those already in the t_{2g} orbitals. Thus there would be parallel spins on the two cations i.e. ferromagnetic coupling between them. These and many other mechanisms have been discussed in detail by Goodenough [26].

All the foregoing discussion has been limited to materials in which the cation-anion-cation angle is 180°. There is, however, a large number of antiferromagnetic compounds, such as those having spinel and corundum crystal structures, in which this angle differs from 180° and, in some cases, is 90°. Simple theories based on overlap between p and d orbitals will not account satisfactorily for the antiferromagnetic exchange in such materials and a more sophisticated theoretical model is required. This has been provided by Halpern [32] who considers the possibility of $s-p$ hybrid orbitals on the anion taking over the role played by the p-orbitals in the simpler compounds. By combining s and p wave functions one obtains a hybrid wave function, the angular dependent part of which will have a maximum in a direction dependent on the relative proportions of the s and p wave functions combined. For example, the sp^3 hybrid, which combines a spherical s-orbital with the three p orbitals in the x, y and z directions has a maximum in a [111] direction as shown in Fig. 4.16. If each anion is surrounded by only four cations, as on

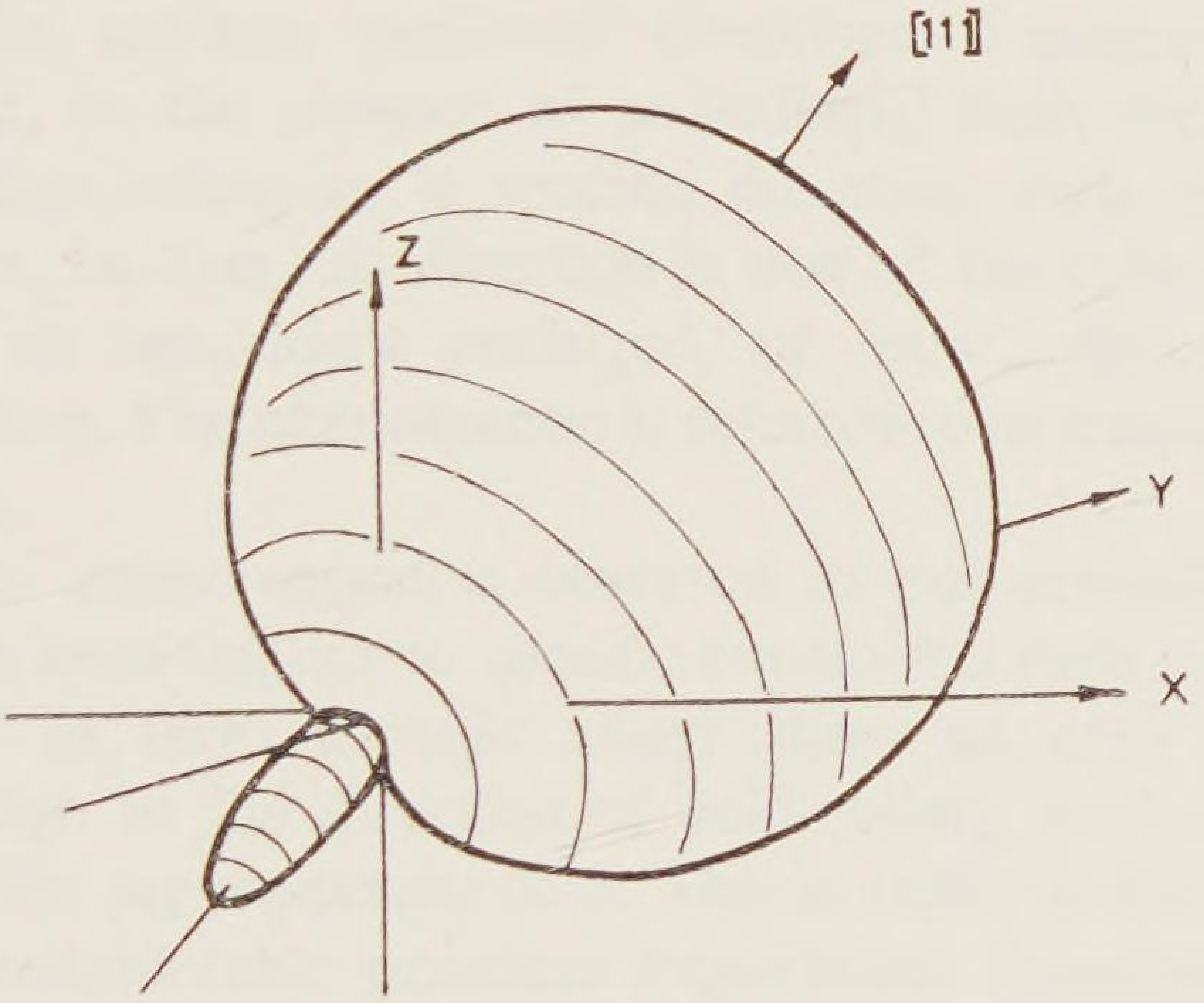


FIG. 4.16

the spinel, corundum and ilmenite crystal structures, it will be possible to find four mutually orthogonal hybrids of the anion *s* and *p* orbitals such that each hybrid is directed principally towards just one cation. Each such directed orbital will contain two electrons, and of the pair of bond orbitals that can be formed from it one will be more strongly combined than the other, as in the simple case discussed above. If these orbitals, from the anion, have an appreciable magnitude in the region of the cation, superexchange interaction between cations can occur.

CHAPTER 5

Magnetic Anisotropy

Introduction

As was mentioned in Chapter 3 a single crystal of magnetic material, such as iron, exhibits 'preferred' directions of magnetization. This means that, in the absence of an external field, the spontaneous magnetization takes up a specific direction with respect to the crystal axes. In iron this direction is one of the cube edges so that there are six equivalent preferred, or 'easy', directions for the magnetization. The phenomenon is referred to as magnetocrystalline anisotropy.

A similar phenomenon is observed in polycrystalline materials which have been subject to specific treatments such as cold-rolling. A number of ferromagnetic alloys show an easy direction for magnetization in the direction of cold-rolling, and some have an easy direction perpendicular to it. This is called induced anisotropy and is of considerable practical importance. Another form of induced anisotropy is found in thin films, particularly of Ni–Fe alloy (Permalloy). If the film is deposited, on to a substrate, by evaporation in vacuum with a magnetic field applied in the plane of the substrate, the direction of the field is subsequently found to be an easy direction for the spontaneous magnetization. This phenomenon is the basis of the use of thin magnetic films as storage elements in computers (see Chapter 8).

We will now consider the theoretical background to magnetic anisotropy.

Anisotropy Energy

When the spontaneous magnetization exhibits a preferred direction this is evidently a minimum-energy position for it. Thus we may argue that, if the magnetization is rotated out of this direction by the application of a field, there must be an increase of energy, in the material, which we describe as the anisotropy energy. In absolute terms we may write down an expression for the total free energy of a crystal as a sum of a number of terms. These terms include the usual thermodynamic ones of enthalpy and entropy and, for a magnetic crystal, also the magnetic energy. The anisotropy energy is a further term which will vary with the angle, θ, by which the magnetization deviates from a preferred direction, being zero for $\theta = 0$. The simplest view, however, is to regard this as the energy which must be supplied in order to rotate the magnetization through the angle θ.

On the basis of the above ideas we may derive expressions for the magnetic anisotropy for various cases. The most obvious one is that of uniaxial anisotropy in which there is a single easy direction of magnetization, such as in single crystal cobalt. This is a hexagonal crystal and the easy direction corresponds to the *c*-axis. The anisotropy energy will be the same whether the magnetization is initially pointing upwards or downwards along this axis and the energy must therefore be some even function of the angle θ. Conventionally it is represented by a power series in $\sin\theta$, including only even terms, i.e.

$$E = K_{u1}\sin^2\theta + K_{u2}\sin^4\theta + \dots \tag{5.1}$$

where the coefficients K_{u1}, K_{u2}, etc. are the magnetocrystalline anisotropy constants. The energy is generally expressed as joules (metre)$^{-3}$ or ergs per cc and it is usually sufficient, for practical purposes, to use only the first term. The anisotropy constant is then simply written as K_u and, for cobalt, is $4{\cdot}1 \times 10^5$ joules/m^3 ($4{\cdot}1 \times 10^6$ ergs/cc). However, it is worth noting that, in cobalt, $K_{u2} = 2 \times 10^5$ joules/m^3.

In the more general case the direction of magnetization is referred to three axes, not necessarily orthogonal, by the direction cosines α_1, α_2 and α_3. Taking the simple case of a cubic crystal, for which the axes are orthogonal, with easy directions along the cube edges (iron), we must again take account of the fact that reversal of the direction of magnetization does not change the anisotropy energy, so that the energy expression must include only even powers of the α's. In addition the expression must be invariant in any interchange of two α's, for example, because the anisotropy energy when the magnetization makes an angle θ to the x-axis in the x-y plane must be the same as when it makes an angle θ to the y-axis in the same plane. In the first case $\alpha_1 = \cos\theta$, $\alpha_2 = \sin\theta$, $\alpha_3 = 0$ and in the second $\alpha_1 = \cos\theta$, $\alpha_2 = 0$, $\alpha_3 = \sin\theta$.

In general, any term incolving $\alpha_i^p\alpha_j^q\alpha_k^r$ must have the same coefficient for any given combination of p, q and r for any interchange of i, j or k.

These conditions will be met, up to sixth order, if we write

$$\begin{aligned} E = {} & A(\alpha_1^2+\alpha_2^2+\alpha_3^2)+B(\alpha_1^4+\alpha_2^4+\alpha_3^4)+C(\alpha_1^2\alpha_2^2+\alpha_2^2\alpha_3^2+\alpha_3^2\alpha_1^2) \\ & +D(\alpha_1^6+\alpha_2^6+\alpha_3^6)+E(\alpha_1^4\alpha_2^2+\alpha_2^4\alpha_3^2+\alpha_3^4\alpha_1^2+\alpha_2^4\alpha_1^2+\alpha_3^4\alpha_2^2+\alpha_1^4\alpha_3^2) \\ & +F(\alpha_1^2\alpha_2^2\alpha_3^2). \end{aligned} \tag{5.2}$$

This may be simplified by use of the usual relationship for direction cosines, i.e.

$$\alpha_1^2+\alpha_2^2+\alpha_3^2 = 1.$$

Since we require that the energy varies with angle, any constant terms are irrelevant, and may be omitted. Thus the term with the coefficient A in (5.2) is omitted. For the B term we have

$$\begin{aligned} \alpha_1^4+\alpha_2^4+\alpha_3^4 &= (\alpha_1^2+\alpha_2^2+\alpha_3^2)^2-2(\alpha_1^2\alpha_2^2+\alpha_2^2\alpha_3^2+\alpha_3^2\alpha_1^2) \\ &= 1-2(\alpha_1^2\alpha_2^2+\alpha_2^2\alpha_3^2+\alpha_3^2\alpha_1^2) \end{aligned}$$

and for the D term we have

$$\begin{aligned} \alpha_1^6+\alpha_2^6+\alpha_3^6 &= (\alpha_1^2+\alpha_2^2+\alpha_3^2)^3 \\ &\quad -3(\alpha_1^4\alpha_2^2+\alpha_2^4\alpha_3^2+\alpha_3^4\alpha_1^2+\alpha_2^4\alpha_1^2+\alpha_3^4\alpha_2^2+\alpha_1^4\alpha_3^2) \\ &= 1-(\alpha_1^2+\alpha_2^2+\alpha_3^2)3(\alpha_1^2\alpha_2^2+\alpha_2^2\alpha_3^2+\alpha_3^2\alpha_1^2)+3\alpha_1^2\alpha_2^2\alpha_3^2 \\ &= 1-3(\alpha_1^2\alpha_2^2+\alpha_3^2+\alpha_3^2\alpha_1^2)+3\alpha_1^2\alpha_2^2\alpha_3^2 \end{aligned}$$

and, similarly,

$$\begin{aligned} & \alpha_1^4\alpha_2^2+\alpha_2^4\alpha_3^2+\alpha_3^4\alpha_1^2+\alpha_2^4\alpha_1^2+\alpha_3^4\alpha_2^2+\alpha_1^4\alpha_3^2 \\ & \quad = (\alpha_1^2+\alpha_2^2+\alpha_3^2)(\alpha_1^2\alpha_2^2+\alpha_2^2\alpha_3^2+\alpha_3^2\alpha_1^2)-3\alpha_1^2\alpha_2^2\alpha_3^2 \\ & \quad = (\alpha_1^2\alpha_2^2+\alpha_2^2\alpha_3^2+\alpha_3^2\alpha_1^2)-3\alpha_1^2\alpha_2^2\alpha_3^2. \end{aligned}$$

Thus all the sixth-order terms can be reduced to a single fourth-order one, together with the sixth-order term in $\alpha_1^2\alpha_2^2\alpha_3^2$. Equation (5.2) may therefore be rewritten, still obeying the conditions specified, as

$$E = K_1(\alpha_1^2\alpha_2^2+\alpha_2^2\alpha_3^2+\alpha_3^2\alpha_1^2)+K_2(\alpha_1^2\alpha_2^2\alpha_3^2) \tag{5.3}$$

where K_1 and K_2 are the first and second anisotropy constants of the material. Typical values, at room temperature are,

for iron $K_1 = 4{\cdot}8\times 10^4$ joules/metre^3 $(4{\cdot}8\times 10^5$ ergs/cc)

$K_2 = \pm 5\times 10^3$ joules/metre^3 $(\pm 5\times 10^4$ ergs/cc)

and for nickel $K_1 = -4{\cdot}5\times 10^3$ joules/metre^3 $(-4{\cdot}5\times 10^4$ ergs/cc)

$K_2 = -2{\cdot}38\times 10^3$ joules/metre^3 $(-2{\cdot}38\times 10^4$ ergs/cc).

Anisotropy Field

Phenomenologically the existence of preferred directions of magnetization, in a single crystal, may be thought of in terms of an 'anisotropy field'. This field may be thought of as an internal one which is directed along each easy axis to maintain the magnetization in this direction. The effect of an external field is then considered to be modified by the anisotropy field so that, for an external field applied at a small angle to the easy axis, the magnetization vector will take up a position determined by the vector resultant of the applied field and the anistropy field. Likewise, to rotate the magnetization from an 'easy' to a 'hard' direction, there would be a coercive field required which would, in theory, be equal to the anisotropy field.

We may relate the anisotropy energy density E to the anisotropy field by relating the magnetization vector to the symmetry of the crystal by means of generalized direction cosines.

(1) *Cubic crystal.* We take the line OX, Fig. 5.1, as being the easy direction to be considered and assume that the magnetization, M, is rotated by a small angle, ϵ, towards another easy direction of the same type, so that it is in the direction OY. There will be a restoring torque on the magnetization, tending to turn it to the position OX, given by $\partial E_k/\partial\epsilon$, where E_k is the anisotropy energy density. If we

treat the anisotropy as if it were a field $\mathbf{H}_k$ directed along OX, then the restoring torque would be given by $MH_k \sin \epsilon$, and by taking

$$\frac{\partial E_k}{\partial \epsilon} = MH_k\epsilon \tag{5.4}$$

for small ϵ we obtain an expression for H_k in terms of M.

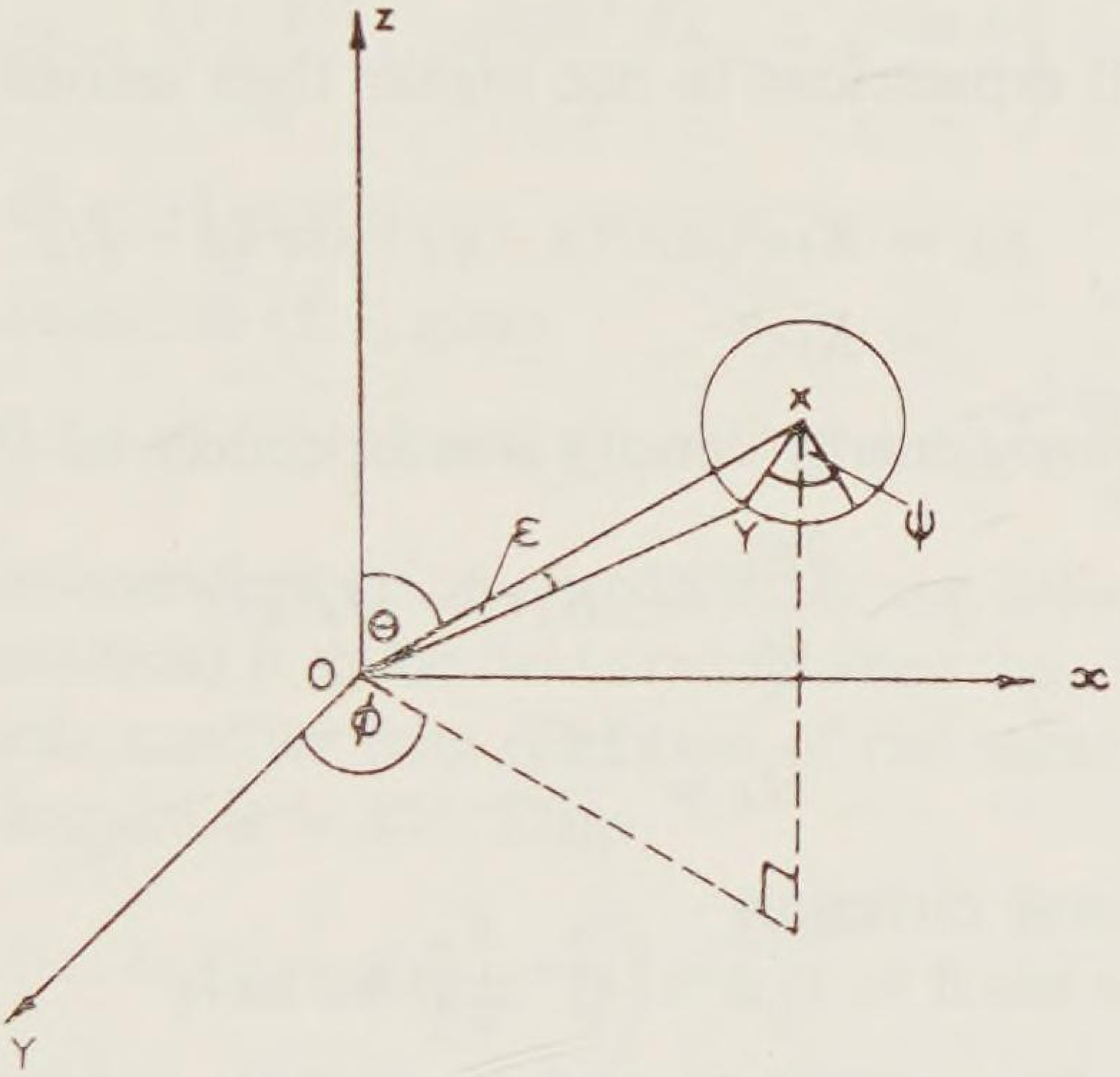


FIG. 5.1

The general direction cosines for the case of Fig. 5.1 have been given by Birss [33] as

$$\left.\begin{aligned}\alpha_1 &= \cos \epsilon \sin \theta \cos \phi + \sin \epsilon(\cos \theta \cos \phi \cos \psi + \sin \phi \sin \psi)\\ \alpha_2 &= \cos \epsilon \sin \theta \sin \phi + \sin \epsilon(\cos \theta \sin \phi \cos \psi - \cos \phi \sin \psi)\\ \alpha_3 &= \cos \epsilon \cos \theta - \sin \epsilon \sin \theta \cos \psi\end{aligned}\right\} \tag{5.5}$$

We approximate $\cos \epsilon = 1-(\epsilon^2/2)$ and $\sin \epsilon = \epsilon$ and evaluate equation (5.3) to obtain the expression for E_k in terms of ϵ, θ, ϕ and ψ. This is a somewhat cumbersome calculation but since, in actual cubic crystals, the magnetocrystalline anisotropy is such that the easy directions are of ⟨100⟩, ⟨110⟩ or ⟨111⟩ types only, the calculation can be simplied by considering these specific directions.

(*a*) ⟨001⟩ easy direction

In this case $\theta = 0°$ and we have

$$\alpha_1^2 = \epsilon^2(\cos\phi\cos\psi+\sin\phi\sin\psi)^2$$
$$\alpha_2^2 = \epsilon^2(\sin\phi\cos\psi-\cos\phi\sin\psi)^2$$
$$\alpha_3^2 = \left(1-\frac{\epsilon^2}{2}\right).$$

Restricting all expressions to not higher than second order in ϵ we obtain

$$E_k = K_1\epsilon^2[\cos^2(\phi-\psi)+\sin^2(\phi-\psi)]$$
$$= K_1\epsilon^2$$

i.e. the anisotropy energy density is independent of ϕ and ψ, and

$$\frac{\partial E_k}{\partial\epsilon} = 2K_1\epsilon = MH_k\epsilon$$

i.e. $$H_k = \frac{2K_1}{M} \tag{5.6}$$

(*b*) ⟨111⟩ easy direction

In this case $\cos\theta = \frac{1}{3}$, $\phi = 45°$ and we have

$$\alpha_1^2 = \frac{(1-\epsilon^2)}{3}+\epsilon^2\left(\frac{\cos^2\psi}{6}+\frac{\sin^2\psi}{2}+\frac{\sin 2\psi}{2\sqrt{3}}\right)+2\epsilon\left(\frac{\cos\psi}{6}+\frac{\sin\psi}{12}\right)$$

$$\alpha_2^2 = \frac{(1-\epsilon^2)}{3}+\epsilon^2\left(\frac{\cos^2\psi}{6}+\frac{\sin^2\psi}{2}-\frac{\sin 2\psi}{2\sqrt{3}}\right)+2\epsilon\left(\frac{\cos\psi}{3\sqrt{2}}-\frac{\sin\psi}{6}\right)$$

$$\alpha_3^2 = \frac{(1-\epsilon^2)}{3}+\frac{2\epsilon^2}{3}\cos^2\psi-\frac{2\sqrt{2}}{3}\epsilon\cos\psi.$$

Using these in equation (5.3) gives

$$E_k = K_1(\tfrac{1}{3}-\tfrac{2}{3}\epsilon^2)+K_2(\tfrac{1}{27}-\tfrac{2}{9}\epsilon^2),$$

which is independent of ψ,

and $$\frac{\partial E_k}{\partial\epsilon} = \tfrac{4}{3}-K_1\epsilon-\tfrac{4}{9}K_2\epsilon = MH_k\epsilon$$

whence $$H_k = -\frac{4}{3M}(K_1+K_2/3). \tag{5.7}$$

Since, for a $\langle 111\rangle$ easy direction, K_1 is always negative and $|K_1|>|K_2|$, H_k is positive.

(*c*) $\langle 110\rangle$ easy direction

In this case $\theta = 90^\circ$ and $\phi = 45^\circ$;

$$\alpha_1^2 = \frac{(1-\epsilon^2)}{2}+\epsilon^2\left(\frac{\sin^2\psi}{2}\right)+2\epsilon\left(\frac{\sin\psi}{2}\right)$$

$$\alpha_2^2 = \frac{(1-\epsilon^2)}{2}+\epsilon^2\left(\frac{\sin^2\psi}{2}\right)-2\epsilon\left(\frac{\sin\psi}{2}\right)$$

$$\alpha_3^2 = \epsilon^2\cos^2\psi.$$

Using these in equation (5.3) gives

$$E_k = K_1\tfrac{1}{4}+\epsilon^2(-\tfrac{1}{2}-\tfrac{1}{2}\sin^2\psi+\cos^2\psi)+K_2\frac{\epsilon^2\cos^2\psi}{4}$$

which is not independent of ψ. In defining the anisotropy field, however, we considered it as the field arising when the magnetization is rotated towards another easy direction of the same type, which means that we require $\psi = 45^\circ$. Then

$$E_k = K_1\left(\frac{1}{4}-\frac{\epsilon^2}{4}\right)+\frac{K_2\epsilon^2}{8}$$

and
$$H_k = \frac{1}{2M}(-K_1+K_2/2). \tag{5.8}$$

The implication of the above is that the effective anisotropy field, in a cubic crystal with a $\langle 110\rangle$-type easy direction, depends upon the direction in which the magnetization is moved away from the easy direction.

(2) *Uniaxial crystals*. This is straightforward with the magnetization making a small angle ϵ with the easy direction. Thus we have

$$E_k = K_u\sin^2\theta = K_u\epsilon^2$$

for small ϵ. The torque equation is therefore

$$2K_u\epsilon = M_sH_k\epsilon$$

i.e.
$$H_k = \frac{2K_u}{M_s}. \tag{5.9}$$

Further references will be made to the concept of anisotropy field in discussing ferromagnetic resonance and magnetic thin films.

Measurement of Anisotropy Constants

(1) *The torque magnetometer.* If a disc, cut from a single crystal, is suspended in a field and magnetized to saturation, so that the magnetization lies everywhere in the *same* easy direction, then the disc will rotate so that this easy direction is aligned with the field. In other words a torque is exerted on the disc, causing it to rotate, and this would be equal to the torque which would be acting on the magnetization vector, causing it to rotate into alignment with the field, if the disc were held clamped. We have seen that this torque on the magnetization, per unit volume L, is given by

$$L = -\frac{\partial E_k}{\partial \theta}. \tag{5.10}$$

and therefore the torque *on the crystal* will be $+\partial E_k/\partial \theta$ where E_k is the anistropy energy density (see also eqn. 2.2). Thus, by measuring the torque exerted on the disc, $\partial E_k/\partial \theta$ can be determined and hence the anisotropy constant. The instrument used is known as a torque magnetometer and is shown schematically in Fig. 5.2. In practice

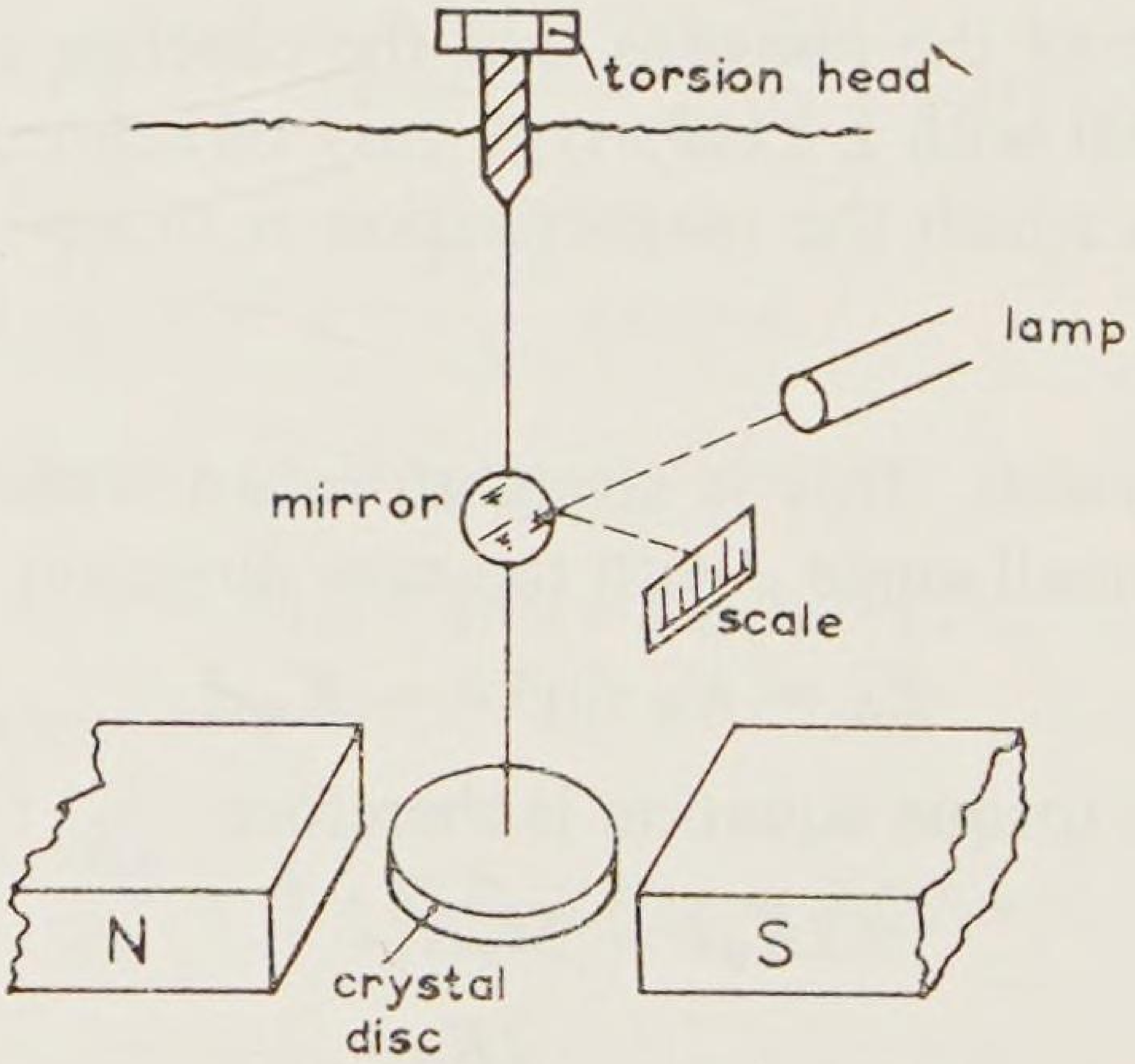


FIG. 5.2